MATHEMATICS AND PUBLIC POLICY - PROBLEMS CONSIDERED IN A FIRST YEAR MATHEMATICS COURSE AT WEST POINT

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The United States Military Academy (USMA) is America's oldest military academy, located on the western bank of the Hudson River about an hour's drive north of New York City. Graduating approximately one thousand Army officers every year, the Academy's students (or cadets, as they are called) take a rigorous academic load in addition to their physical and military training. As part of the overall core curriculum, the first semester mathematics course taken by the majority of freshmen is called MA103: Mathematical Modeling and Introduction to Calculus. As its name suggests, it gives students ample opportunity to model and solve real-world problems with mathematics, many of which have important policy implications. I taught this course in the fall of 2014, and this paper discusses four of the many modeling problems that were used.

Introduction to Mathematical Modeling

A mathematical model is "a construct (e.g. a function, equation(s)) that is designed to predict the behavior of a system".² Models are used every day in different disciplines to help people make informed decisions. Whether you are trying to estimate life expectancy, weather patterns, or even how much to charge for a good or service, mathematical models can be quite useful.





¹The views expressed herein are those of the author and do not reflect the position of the United States Government, the United States Army, or the United States Military Academy.

² Modeling in a Real and Complex World", Chapter 1.1.1., Department of Mathematical Sciences, United States Military Academy, 2014.



Figure 1: The Mathematical Modeling Triangle

An introduction to mathematical modeling is taught in the first year at West Point to emphasize the problem-solving process. Since all graduates will go on to become commissioned officers in the United States Army, it is important that these leaders can not only solve problems they have encountered before, but can solve new or ill-defined problems that they may encounter in the future. For this reason, we emphasize a general process to demonstrate that a good problem-solving process can be adapted to almost any situation. In this course, we use the mathematical modeling triangle³ (Figure 1).

We further emphasize that mathematical modeling is more than coming up with a feasible solution to a problem. Rather, it is iterative in nature, with continuous refining of the model. In MA103, we take students through a series of steps designed to engage them in the problem-solving process. First, they *transform* their problem into a mathematical model, then they *solve* using the necessary tools and steps, and finally, they *interpret* their solution. As Figure 1 demonstrates, the process is iterative, and can consist of many cycles in order to refine a model and solution.

The following four examples give a cross-section of problems addressed in MA103 that challenge students to think not only of the best way to model a problem, but also of the policy implications of their assumptions and solution.

³"Modeling in a Real and Complex World", Figure 1.4., Department of Mathematical Sciences, United States Military Academy, 2014.

Modeling Atmospheric Concentration of CO2

Problem: Given a set of 20th century atmospheric CO2 concentration data collected in the vicinity of the Manau Loa⁴ volcano in Hawaii, model the future atmospheric CO2 concentration as a discrete dynamical system (DDS), and consider the policy implications of the results of your model.

A DDS consists of a recursion equation and an initial condition. It is a discrete model where any future value p_n at time n depends on the previous values p_{n-1} , p_{n-2} , etc. Consider the following general model of a first-order, linear, non-homogeneous DDS, where the initial condition is given by p_0 :

$$\begin{cases} p_n = a \, p_{n-1} + d \\ p_0 \end{cases}$$

Several questions that were asked of students were to provide intuition as to what the *a* and *d* parameters represent in their model, use the model for interpolation and extrapolation, predict the future CO2 atmospheric concentration past 2014, and provide a policy recommendation based on the results of their model.⁵ We first asked students to model the increase in CO2 as a result of human industrial activity and natural processes (e.g. forest fires etc.) as a constant with the *d* parameter, and the decrease in CO2 as a result of naturally occuring processes⁶ (carbon sinks such as oceans, photosynthesis, etc.) with the *a* parameter. By fitting the first and last points of the actual data, we come up with the following model:

$$\begin{cases} p_n = 0.97 \, p_{n-1} + 12.5 \\ p_0 = 315.97 \end{cases}$$



 $^{^4}$ National Oceanic and Atmospheric Administration (NOAA), U.S. Department of Commerce, Earth System Research Laboratory, "Trends in Atmospheric Carbon Dioxide," available at http://www.esrl.noaa.gov/gmd/ccgg/trends/mlo.html#mlo_data

⁵ Policymakers' Summary," in Climate Change: The IPCC Scientific Assessment (Cambridge: Cambridge University Press, 1990), p. XVIII.

 $^{^{6}\}mathrm{NASA}, \mathrm{http://science.nasa.gov/earth-science/oceanography/ocean-earth-system/ocean-carbon-cycle/$



Figure 2: Time-series scatter plot of the actual concentration of CO2 and the predicted concentration generated from the recursion equation $p_n = 0.97 p_{n-1} + 12.5$

The long-term behavior of the model is converging to some equilibrium point, whereas the data suggests that the long-term behavior is diverging (Figure 3). By following the modeling process (Figure 1), we update the model by modeling the increase in CO2 with the *a* parameter (CO2 increases by a proportion of current concentration), and any decrease with the *d* parameter (CO2 decrease is constant):

$$\begin{cases} p_n = 1.0181 \, p_{n-1} - 4.813 \\ p_0 = 315.97 \end{cases}$$







The long-term behavior of the second model is diverging (Figure 3). In other words, without some sort of change to the current conditions or parameters, the model suggests that the atmospheric concentration of CO2 will not approach an equilibrium and will continue to rise. Students had to weigh the real-life implications of their model, and discuss both the validity of their assumptions, and the feasibility (economic, environmental, political, engineering) of implementing policies based on their prediction of future values.

Modeling Population Growth

Problem: Given a current population profile, use a Leslie Matrix to determine the long-term behavior of the population.

A Leslie matrix corresponding to a population model with i age groups is an $(i \times i)$ matrix, A, whose entries represent the reproduction and survival rates of each group, as follows:⁷

$$A = \begin{bmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,i-1} & a_{1,i} \\ a_{2,1} & 0 & 0 & \dots & 0 & 0 \\ 0 & a_{3,2} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{i,i-1} & a_{i,i} \end{bmatrix}$$

The first row of A contains reproduction rates and the remaining rows contain survival rates. The matrix is sparse since most of the entries are zero (reflecting the assumption that survival rates are only positive between adjacent population groups). A system of first-order, linear, homogeneous recursion equations is given by the following:

$$\vec{p}_n = A \, \vec{p}_{n-1}$$

 \vec{p}_n is the vector of populations of *i* groups in year *n*.





⁷"Modeling in a Real and Complex World", Chapter 3.3. Department of Mathematical Sciences, United States Military Academy, 2014.



Figure 4: 2015 Population Pyramid of India

The reproduction and survival rates can be estimated using census data, such as this population pyramid from India⁸ (Figure 4). These models can be quite useful, from predicting the number of young and adult fish in a body of water to the future population of a country. Each solution derived from a student's model has important implications to consider, such as the effect and sustainability of current policies and practices.

Modeling the future STEM workforce in the United States

Problem: Given a set of data including annual high school graduation numbers⁹, entrance rates into STEM majors¹⁰, graduation rates¹¹, and percentages of graduates entering the STEM workforce¹², recommend a policy with respect to the need for Science, Technology, Engineering, and Mathematics (STEM) outreach among the high school population.

With the given data, students modeled the population each year in three key groups (College freshmen, college graduates in a STEM discipline, new entrants into the STEM workforce) using a system of first-order, linear, non-homogeneous recursion equations where:

$$\vec{p_n} = A\,\vec{p_{n-1}} + \vec{d}$$

 \vec{p}_n is the vector of populations from the three key groups in year n. A is a matrix that is comprised of the rates of passage of each group from year n-1 to year n.

⁸United States Census Website, http://www.census.gov, 2015.

 $^{^{9}}$ National Center for Education Statistics, http://nces.ed.gov/programs/coe/indicator_coi.asp 10 National Center for Education Statistics, http://nces.ed.gov/pubs2014/2014001rev.pdf 11 Ibid.

¹²United States Census Bureau, http://www.census.gov/dataviz/visualizations/stem/stem-html/

d is a constant vector that is added or subtracted to each group from year n-1 to year n.

Students made reasonable and necessary assumptions to develop their model, then discussed their findings and implications with regards to STEM outreach efforts. It was generally hypothesized, according to their models, that while the number of high school graduates was increasing each year, the number of students graduating with STEM degrees and entering in the STEM workforce was declining, strengthening the argument in favor of STEM outreach effort targeting the high school population.

Modeling Heat Exchange in Soldiers' Tents in Afghanistan

Problem: 2009 field data from a Unites States Marine Corps study in Afghanistan concluded that for every gallon of generator fuel used, it took seven gallons to transport it there.¹³ At the tactical level, power demand was small, and that generators used to cool tents in the heat of summer comprised the majority of power demand.¹⁴ The military began using a spay foam to coat the tents in order to decrease energy consumption by the generators, saving the government money.¹⁵ This also caused the need for fewer fuel convoys, potentially saving lives in the process by exposing Soldiers to fewer enemy attacks.

Students were asked to model the heat exchange between the ambient outside temperature and the inside temperature of tents that are coated in foam (Figure 5), and to determine the on-off cycle needed to maintain the temperature inside the tents within a given range.



Figure 5: Tents in Afghanistan using Spray Foam

¹³Marine Energy Assessment Team, 1 October 2009.

 $^{^{14}}$ Ibid., 1 October 2009.

¹⁵Sprayfoam.com. http://www.sprayfoam.com/newsarchives/archivedetails.cfm?id=418, April 2015.

Students were given a system of two recursion equations with a single parameter k to model the daytime temperature inside and outside of the tents in a given hour n, where:

$$\vec{p}_n = \left[\begin{array}{cc} 1 & 0\\ k & 1-k \end{array} \right] \vec{p}_{n-1}$$

 \vec{p}_n is a vector of outside (O_n) and inside temperatures (I_n) at hour n. k is a parameter between 0 and 1 which is dependent on the amount of foam used (lower k value means more foam is used)

Students determined that as the k parameter increased, the energy consumption increased, requiring more work from the air conditioner units and more fuel. They were then asked to determine the best on-off cycle for a specified k parameter to keep the temperature inside of the tents below 30 degrees Celsius. The initial outside temperature (O_0) was given as 40 degrees Celsius and the initial inside temperature (I_0) was given as 20 degrees Celsius. Modeling real-world problems such as this allowed students to consider how mathematics can be used to inform policy decisions at both local and higher levels.

Conclusion

The first semester core mathematics course taught at the United States Military Academy is primarily a modeling course, built to promote problem-solving and critical thinking. The various problems that are addressed in a classroom environment expose students to mathematical problems with important policy implications, allowing them to challenge their assumptions and make educated recommendations based on quantitative analysis. This supports the West Point mission to provide graduates who are critical thinkers, adept at solving problems, and who are capable of thriving in a highly complex world.



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