

ILLUSTRATING THE EXTENSION OF A SPECIAL PROPERTY OF CUBIC POLYNOMIALS TO NTH DEGREE POLYNOMIALS

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Introduction

Miller (2011) discussed how to use the free interactive algebra-geometry program Geogebra, to investigate a special property of cubic polynomials. The article demonstrated various ways to use Geogebra to illustrate the special property that *given a cubic polynomial with three real zeros, a tangent drawn to the curve at the point at which the abscissa is the mean of two of those three zeros will intersect the horizontal axis at the other zero*. This property was extended by Miller and Moseley (2012) to higher degree polynomials. We will use Geogebra to illustrate how this special property of cubic polynomials can be generalized to higher degree polynomials. This will be done by first reviewing the cubic polynomial case through an applet, discussing and illustrating in Geogebra the cases for fourth and fifth degree polynomials along with derivations of formulas. The proof for the 4th degree polynomial case, and the general proof for the n th degree polynomial with zeros x_1, x_2, \dots, x_n is shown in (Miller and Moseley, 2012).

Cubic Polynomials

Miller (2011) uses Geogebra to present a variety of ways to illustrate different examples of the special property for cubic polynomials. Instead of reiterating what is everything described about cubics in Miller (2011) and Miller and Moseley (2012), we will construct an applet in Geogebra to illustrate the case where all the zeros of the cubic polynomial are real numbers and then state the proof. First go to www.geogebra.org and click on the download tab that appears between “About” and “Help” near the upper right corner. Download a free version of Geogebra by clicking on the Applet Start button. Now follow the steps below and refer to Miller (2011) under the subheading “Adding More Visual Proof” for more detailed instructions.

1. Insert three sliders by clicking on the slider menu and clicking in the graphing window three separate times (see figure 1 below). Label the sliders a, b, and c with minimum -10, maximum 10, and increment of 0.01.



Figure 1: Slider Button

2. In the input box (bottom of Geogebra) type in

$$f(x) = (x-a)*(x-b)*(x-c).$$

Your screen should look like figure 2 below.

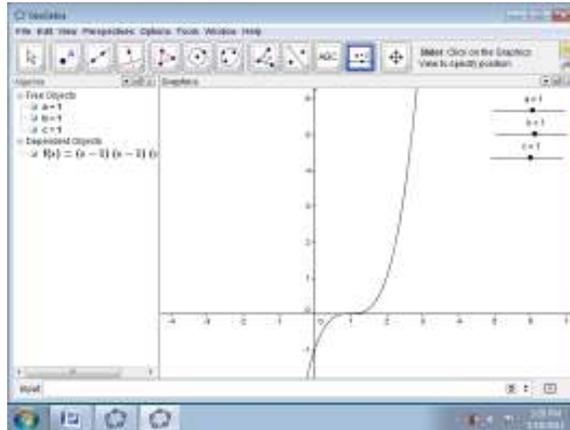


Figure 2: General Cubic with all zeros at 1.

- Adjust a to be 2, b to be 1, and c to be -1. Your screen should look similar to Figure 3 below.

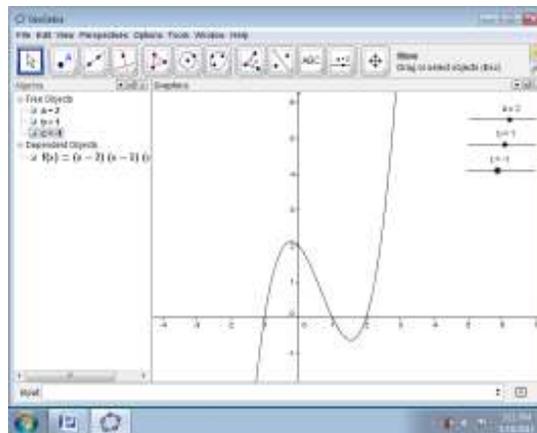


Figure 3: Cubic with Specified Sliders

- Now we need to capture the x-intercepts. Start typing in roots and when the input box gives you a drop down menu of different commands with root in them, select the command **Roots[<Function>, <Start x-Value>, <End x-Value>]** and insert $f(x)$ for <Function>, -10 for <Start x-value>, 10 for <End x-Value>, and Enter. You should get points A, B, and C for the x-intercepts of the polynomial. Your screen should look like figure 4 below.
- Input into the input bar, **$g(x) = \text{Derivative}[f(x)]$** . Click on the radio button next to $g(x)$ in the algebra box (under the dependent objects) to hide it.
- Type in $d = (a+b)/2$ into the input box. Repeat by typing $e=(a+c)/2$ and $h=(b+c)/2$.

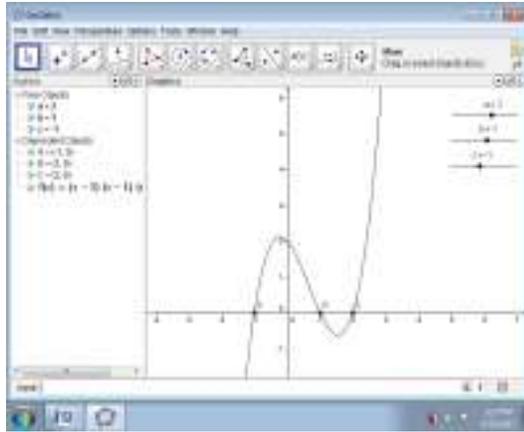


Figure 4: Cubic with Marked Zeros

7. Now type in to the input box separately, $y - f(d) = g(d) \cdot (x - d)$, $y - f(e) = g(e) \cdot (x - e)$, and $y - f(h) = g(h) \cdot (x - h)$. Your screen should look similar to Figure 5 below.

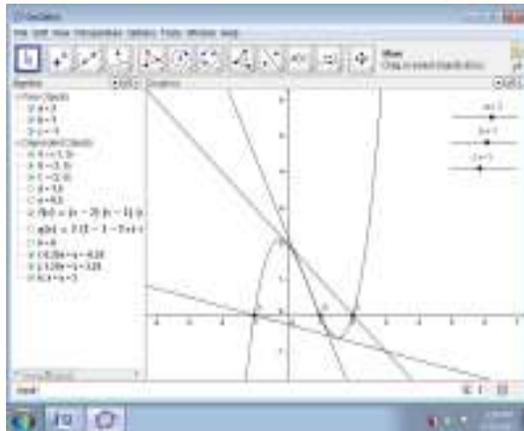


Figure 5: Cubic with Tangent Lines

8. Polish up the screen by inserting color, line thickness, by doing the following:
 - Right click on $f(x)$ and go to object properties. Choose a color (I choose blue) and line thickness (under style) to be 4. Repeat for each tangent line and the sliders.
 - Under object properties \rightarrow basic, you can label $f(x)$ and each tangent line by selecting the box for show label and clicking on the down arrow in the selection box and choosing name & value option. Your screen should look similar figure 6 below.

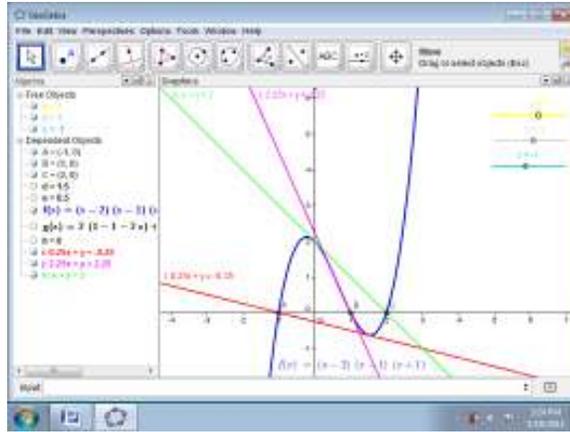


Figure 6: Polished Version of the Cubic Polynomial with Sliders

Notice that as you move the sliders to different values, the tangent line at the average of two of the zeros always has an x-intercept at the other zero. This shows all the different examples for real zeros (even in the trivial case when two or three of the zeros are the same), except when two of the zeros are complex conjugates. The proof below shows why this always works for cubic polynomials with zeros a , b , and c and can be found in (Miller and Moseley, 2012). Students believe that they can say something is true (proved) if they show some examples. Therefore it might be best to emphasize why the proof is necessary.

Given any cubic polynomial, we can factor it by the fundamental theorem of algebra into the form $f(x) = k(x-a)(x-b)(x-c)$ where a, b , and c are complex numbers (notice in the Geogebra applet, $k = 1$. One could insert another slider to incorporate the scaling factor k). Assume that $a \neq b$. Otherwise the derivation is trivial when $a = b$ or $a = b = c$. Evaluating the function at the average of a and b , we have

$$f\left(\frac{a+b}{2}\right) = k\left(\frac{b-a}{2}\right)\left(\frac{a-b}{2}\right)\left(\frac{a+b}{2} - c\right) = -k\left(\frac{b-a}{2}\right)^2\left(\frac{a+b-2c}{2}\right).$$

Evaluating the derivative at the average of a and b , we have

$$f'\left(\frac{a+b}{2}\right) = -k\left(\frac{b-a}{2}\right)^2 + k\left(\frac{b-a}{2}\right)\left(\frac{a+b}{2} - c\right) + k\left(\frac{a-b}{2}\right)\left(\frac{a+b}{2} - c\right) = -k\left(\frac{b-a}{2}\right)^2$$

So the tangent line is

$$y = -k\left(\frac{b-a}{2}\right)^2\left(x - \frac{a+b}{2}\right) - k\left(\frac{b-a}{2}\right)^2\left(\frac{a+b-2c}{2}\right)$$

Substituting $y = 0$ and solving for x we have

$$0 = -k\left(\frac{b-a}{2}\right)^2\left(x - \frac{a+b}{2}\right) - k\left(\frac{b-a}{2}\right)^2\left(\frac{a+b-2c}{2}\right)$$

$$0 = -k\left(\frac{b-a}{2}\right)^2\left[\left(x - \frac{a+b}{2}\right) + \left(\frac{a+b-2c}{2}\right)\right]$$

$$0 = \left(x - \frac{a+b}{2}\right) + \left(\frac{a+b-2c}{2}\right)$$

This last equation simplifies to $0 = x - c$ and so $x = c$, the other zero. Since there was no preference why we picked the zeros a and b to average, we would obtain the same result if we chose any two distinct zeros.

We see an awesome property of cubic polynomials, but we are left with a feeling that there is more to this than what meets the eye. Is there some hidden explanation for why it works? Does the tangent line to an n th degree polynomial at the average of all but one of the zeros always have an x -intercept at the other zero, for $n > 3$? If not, is there another formula in terms of all but one of the zeros, besides averaging, that would yield the same result? Although we do not answer these questions explicitly in this article, the reader can reference (Miller and Moseley, 2012) to get more details of the formulas, underlying property from Calculus, and general proofs. We will concentrate on building Geogebra applets with sliders to illustrate how the cubic property can be generalized to higher degree polynomials and use the formulas in the applets.

Fourth Degree Polynomial Example

Now we will look at building examples of fourth and fifth degree polynomials. Let a , b , c , and d be the real zeros of the fourth degree polynomials (zeros could be complex, but can not be visualized in Geogebra). The formula for fourth degree polynomials $f(x)$ is derived in (Miller and Moseley, 2012). Given three of the zeros of $f(x)$, say a , b , and c , and the solutions, s_1 and s_2 , of the following quadratic equation $3x^2 - 2(a+b+c)x + ab + bc + ac = 0$, the tangent line at $(s_1, f(s_1))$ and $(s_2, f(s_2))$, intersects the x -axis at the other zero d . To illustrate this we will build a Geogebra applet with the following steps.

1. Insert four sliders by clicking on the slider menu and labeling them a , b , c , and d with minimum -10 , maximum 10 , and increment of 0.01 .
2. In the input box (bottom of Geogebra) type in $f(x) = (x-a)*(x-b)*(x-c)*(x-d)$.
Your screen should look figure 7 below.
3. Adjust a to be -2 , b to be -1 , c to be 0 , and d to be 1 . Your screen should look like figure 8 below.

4. Now we need to capture the x-intercepts. Start typing in roots and when the input box gives you a drop down menu of different commands with root in them, select the command **Roots[<Function>, <Start x-Value>, <End x-Value>]** and insert f(x) for <Function>, -10 for <Start x-value>, 10 for <End x-Value>, and Enter. You should get points A, B, C, and D for the x-intercepts of the polynomial. Your screen should look like figure 9 below.
5. Input into the input bar, **$g(x) = \text{Derivative}[f(x)]$** . Click on radio button next to g(x) in the algebra box (under the dependent objects) to hide it.
6. Type in **$e = (a+b+c + \sqrt{a^2+b^2+c^2-(ab+ac+bc)})/3$** and **$h = (a+b+c - \sqrt{a^2+b^2+c^2-(ab+ac+bc)})/3$** into the input box.
7. Now type in **$y-f(e) = g(e)*(x-e)$** to get the tangent line at e that has an x-intercept at d. Repeat for tangent line at h by inputting **$y-f(h) = g(h)*(x-h)$** . Your screen should look like figure 10 below.
8. Polish up the screen by inserting color, and line thickness (see figure 11).

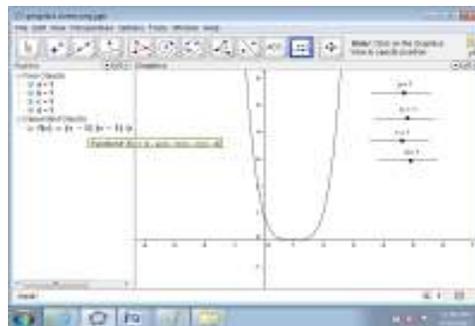


FIGURE 7: Generalized 4th degree polynomial with sliders set at 1

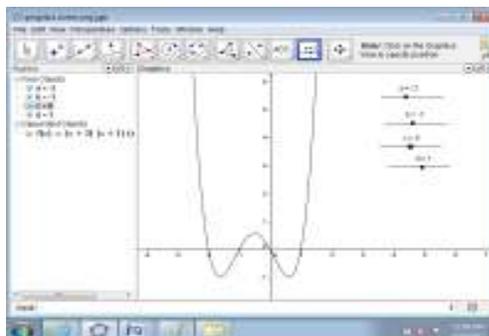


FIGURE 8: Generalized 4th degree polynomial with sliders for the zeros.

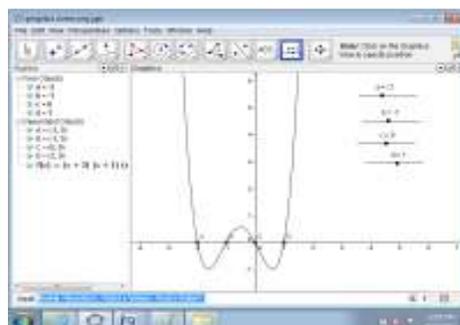


FIGURE 9: Generalized 4th degree with zeros labeled A,B, C, and D

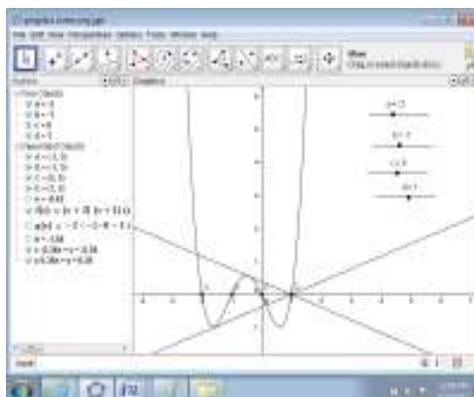


FIGURE 10: Generalized 4th degree polynomial w/ tangent lines that intersect at D

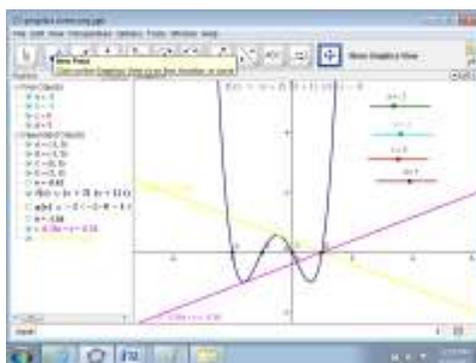


FIGURE 11: Polished Up Version of 4th degree polynomial

It is to note that we are only showing the tangent lines that intersect at d. There are six more tangent lines, two that have x-intercepts at a, two at b, and two at c. The x-values of the point of tangency can be calculated by a formula similar to the formula in 6 above (the formula would be in terms of b,c,d, and e for the tangent lines that intersect the horizontal axis at a). If you would construct all the tangent lines then you would get the follow view iin Geogebra (see figure 12).

Fifth Degree Polynomial Example

Let a , b, c, d, and e be the real zeros of the fifth degree polynomial $f(x)$. The formula for $f(x)$ is derived in (Miller and Moseley, 2012). Given four of the zeros of $f(x)$, say a , b, c, and d, the solutions of the cubic equation

$$4x^3 - 3(a + b + c + d)x^2 + 2(ab + ac + ad + bc + bd + cd)x - (abc + abd + acd + bcd) = 0,$$

say s_1 , s_2 , and s_3 the tangent line at $(s_1, f(s_1))$ and $(s_2, f(s_2))$, and $(s_3, f(s_3))$ intersects the x-axis at the other zero e. To illustrate this we will build a Geogebra applet with the following steps.

1. Input five sliders and label them a, b, c, d, and e. Select Min to be -10, Max to be 10, and Increment to be 0.001.
2. Input into the Input box $f(x) = (x-a)*(x-b)*(x-c)*(x-d)*(x-e)$ and slide the sliders so that a is -2, b is -1, c is 0, d is 1, and e is 2.
3. Type in Roots[$f(x)$, -10, 10]. Your screen should look similar to figure 13 below.

4. Now we want to find the solutions to the (cubic) equation we get from the proof for higher degree polynomials. In (Miller and Moseley, 2012) we see that for higher degree polynomials we have

$$\sum_{i=1}^{n-1} \prod_{\substack{j=1 \\ i \neq j}}^{n-1} (x_0 - x_j) = 0$$

where $x_1, x_2, x_3,$ and x_4 are four of the five zeros of the fifth degree polynomial.

Expanding this out we have

$$(x_0 - x_2)(x_0 - x_3)(x_0 - x_4) + (x_0 - x_1)(x_0 - x_3)(x_0 - x_4) + (x_0 - x_1)(x_0 - x_2)(x_0 - x_4) + (x_0 - x_1)(x_0 - x_2)(x_0 - x_3)$$

Let $x_1 = a, x_2 = b, x_3 = c,$ and $x_4 = d$ (parameters corresponding to the parameters in the applet).

$$(x_0 - b)(x_0 - c)(x_0 - d) + (x_0 - a)(x_0 - c)(x_0 - d) + (x_0 - a)(x_0 - b)(x_0 - d) + (x_0 - a)(x_0 - b)(x_0 - c) = 0$$

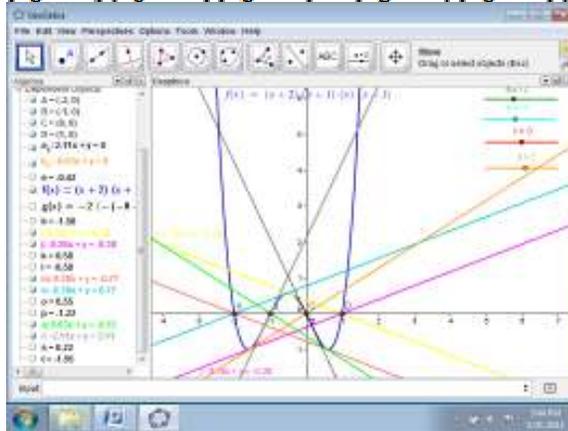


Figure 12: Quartic showing all tangent lines.

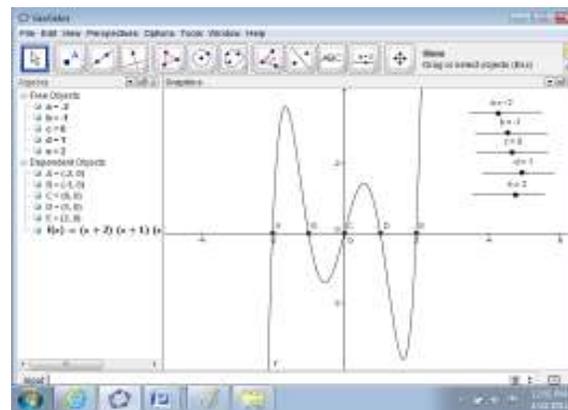


FIGURE 13: 5th degree polynomial with sliders

To solve this we will type in the input box the two commands

$$m(x) = (x - b) * (x - c) * (x - d) + (x - a) * (x - c) * (x - d) + (x - a) * (x - b) * (x - d) + (x - a) * (x - b) * (x - c)$$

and `Roots[m(x), -10, 10]`.

The roots are the three points where the tangent line has an x-intercept at the other zero e. Your screen should look similar to figure 14 below (where the line type (under object properties that you can get to by right clicking on m(x)) was changed to dotted).

5. To find the zeros of m(x), type in **Root[m(x)]**. You should see the points F, G, and H appear on the graph and in the algebra box.
6. These are the three points where the tangent line (at these points) have x-intercepts at e. To see this we first must obtain the x-coordinates of F, G, and H by typing **x(F)**, **x(G)**, and **x(H)**. These will appear in the algebra window as g, h, and I (since f was used for the function).
7. Type **j(x)=Derivative[f(x)]** and hide it.

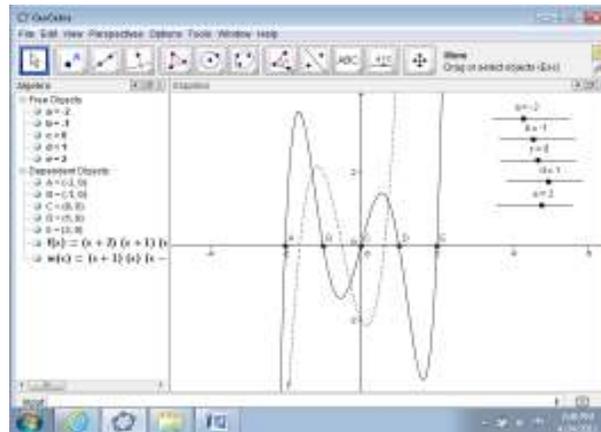


Figure 14: 5th degree polynomial with a dotted derivative

8. Now type **y-f(g)=j(g)*(x-g)**, **y-f(h)=j(h)*(x-h)**, and **y-f(i)=j(i)*(x-i)**.
9. Now polish things up by adding color, line thickness and labeling. Your screen should be similar to figure 15.

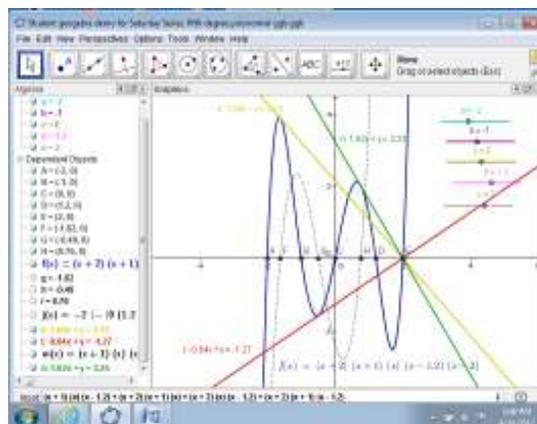


FIGURE 15: Polished up Version 5th degree polynomial

Conclusion

This article has shown how to use Geogebra to extend a mysterious property of cubic polynomials to more general properties for fourth and fifth degree polynomials. In general one can illustrate this for an n th degree polynomial to find the $n-2$ points on the x -axis in terms of $n-1$ zeros, that is given by the solutions to the equation below, where x_1, x_2, \dots, x_{n-1} are the $n-1$ zeros,

$$\sum_{i=1}^{n-1} \prod_{\substack{j=1 \\ i \neq j}}^{n-1} (x - x_j) = 0$$

such that the tangent line to the n th degree polynomial at each $(s_i, f(s_i))$ ($i = 1, 2, \dots, n - 1$) intersects the x -axis at the other zero x_n . Here s_i for $i = 1, 2, \dots, n - 1$ are the zeros of the resulting $n-2$ degree polynomial derived from the formula above. See (Miller and Moseley, 2012) for more details and the underlying calculus concept behind the general property for polynomials. The reader should illustrate this for the 6th degree polynomial and think about the proof before referencing the articles.

This article shows some basic functions of Geogebra so that the reader can familiarize themselves with the program and is a good tool for students to discover some mathematics about polynomials in which they can see some specific examples and work on a more general proof via paper and pencil.

References

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