ON THE *CASUS IRREDUCIBILIS* OF SOLVING THE CUBIC EQUATION

Jay Villanueva
Florida Memorial University
Miami, FL 33055
jvillanu@fmuniv.edu

****

I. Introduction

We often need to solve equations as teachers and researchers in mathematics. The linear and quadratic equations are easy. There are formulas for the cubic and quartic equations, though less familiar. There are no general methods to solve the quintic and other higher order equations. When we deal with the cubic equation one surprising result is that often we have to express the roots of the equation in terms of complex numbers although the roots are real. For example, the equation

$$y^3 - 15y - 4 = 0$$

has all roots real, yet when we use the formula we get

$$y_1 = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}.$$

This root is really 4, for, as Bombelli noted in 1550,

$$(2 + i)^3 = 2 + 11i \quad \text{and} \quad (2 - i)^3 = 2 - 11i\ ,$$

and therefore $(2 + i) + (2 - i) = 4.$ This is one example of the *casus irreducibilis* on solving the cubic equation with three real roots.

II. Cardan's formulas

The quadratic equation $ax^2 + bx + c = 0$, with real coefficients, has the solutions

(1) $$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The discriminant

(2) $$\Delta = b^2 - 4ac = 0, \quad \begin{cases} < 0, & \text{two complex roots} \\ = 0, & \text{real double root} \\ > 0, & \text{two real roots.} \end{cases}$$

The (monic) cubic equation $x^3 + bx^2 + cx + d = 0$, $\quad a = 1$, can be reduced by the transformation $x = y - \frac{b}{3}$ to the form $y^3 + py + q = 0$, where

(3) $$p = -\frac{b^2}{3} + c, \qquad q = \frac{2b^3}{27} - \frac{bc}{3} + d.$$

Using the abbreviations

(4) $$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \quad B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \quad \text{and} \quad \omega = \frac{-1 + i\sqrt{3}}{2},$$

we get Cardans' formulas (1545):

(5) $$\begin{aligned} y_1 &= \sqrt[3]{A} + \sqrt[3]{B} \\ y_2 &= \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B} \\ y_3 &= \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}. \end{aligned}$$

The complete solutions of the cubic are:

(6) $$x_k = y_k - \frac{b}{3}, \qquad k = 1, 2, 3.$$

The roots are characterized by the discriminant

(7) $$\begin{aligned} \Delta &= (y_1 - y_2)^2(y_2 - y_3)^2(y_3 - y_1)^2 \\ &= -27q^2 - 4p^3 \\ &< 0, \quad \text{one real, two complex roots} \\ &= 0, \quad \text{multiple roots} \\ &> 0, \quad \text{three real roots.} \end{aligned}$$

The third case, where $\Delta > 0$, is the *casus irreducibilis*. We note in passing, from the Intermediate Value Theorem, that a cubic (odd degree) polynomial has at least one real root.

III. Examples

Example 1. The cubic $x^3 - 8x - 3 = 0$ is already in reduced form, with $p = -8, q = -3,$ $\Delta = +1805.$ Its roots are given by Eq. (5), with:

$$A = \frac{3}{2} + \sqrt{-\frac{1805}{108}} = \frac{3}{2} + i\,\frac{19}{6}\sqrt{\frac{5}{3}}, \quad \text{and} \quad B = \frac{3}{2} - i\,\frac{19}{6}\sqrt{\frac{5}{3}};$$

$$x_1 = \sqrt[3]{A} + \sqrt[3]{B} = \sqrt[3]{\frac{3}{2} + i\,\frac{19}{6}\sqrt{\frac{5}{3}}} + \sqrt[3]{\frac{3}{2} - i\,\frac{19}{6}\sqrt{\frac{5}{3}}}$$

$$= \frac{1}{2}\left(3 + i\sqrt{\frac{5}{3}}\right) + \frac{1}{2}\left(3 - i\sqrt{\frac{5}{3}}\right) = 3,$$

$$x_2 = \frac{1}{4}(-1 + i\sqrt{3})\left(3 + i\sqrt{\frac{5}{3}}\right) + \frac{1}{4}(-1 - i\sqrt{3})\left(3 - i\sqrt{\frac{5}{3}}\right) = \frac{1}{2}(-3 - \sqrt{5}).$$

$$x_3 = \frac{1}{4}(-1 - i\sqrt{3})\left(3 + i\sqrt{\frac{5}{3}}\right) + \frac{1}{4}(-1 + i\sqrt{3})\left(3 - i\sqrt{\frac{5}{3}}\right) = \frac{1}{2}(-3 + \sqrt{5}).$$

Remark 1: All the roots are real, but they are all expressed in terms of complex quantities.
Remark 2: In this example, we were able to simplify the cube roots of complex numbers in terms of other complex numbers:

$$\sqrt[3]{\frac{3}{2} + i\,\frac{19}{6}\sqrt{\frac{5}{3}}} = \frac{1}{2}\left(3 + i\sqrt{\frac{5}{3}}\right) \quad \text{and} \quad \sqrt[3]{\frac{3}{2} - i\,\frac{19}{6}\sqrt{\frac{5}{3}}} = \frac{1}{2}\left(3 - i\sqrt{\frac{5}{3}}\right).$$

This can be verified by expanding both sides to power 3.

Example 2. The reduced cubic $y^3 - 3y + 1 = 0$ has $p = -3, q = 1, \Delta = 81,$ and so it has real roots. Then

$$A = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2} = \omega \quad \text{and} \quad B = -\frac{1}{2} - i\,\frac{\sqrt{3}}{2} = \omega^2.$$

The roots are $y_1 = \sqrt[3]{\omega} + \sqrt[3]{\omega^2}$

$y_2 = \omega\sqrt[3]{\omega} + \omega^2\sqrt[3]{\omega^2}$

$y_3 = \omega^2\sqrt[3]{\omega} + \omega\sqrt[3]{\omega^2}$.

Again, the roots are all expressed in terms of cube roots of complex numbers, yet all of them are real. Let us attempt to find

$\sqrt[3]{\omega} = a + ib,$ rational a and b to be determined.

This leads to two simultaneous equations

$$a^3 - 3ab^2 = -\frac{1}{2}, \qquad 3a^2b - b^3 = \frac{\sqrt{3}}{2}.$$

Solving for b and b^2, and eliminating, we find

$$(2a)^9 + 3(2a)^6 - 24(2a)^3 + 1 = 0.$$

Setting $x = (2a)^3,$ we have for x a cubic equation

$$x^3 + 3x^2 - 24x + 1 = 0,$$

which, by the transformation $x = y - 1$, becomes

$$y^3 - 27y - 27 = 0,$$

and, setting $y = -3z,$ becomes

$$z^3 - 3z + 1 = 0.$$

This is the same equation we started with before. Thus, we have not advanced in trying to find a and b, as we have found in the previous example.

IV. Significance

We have seen in the examples that a cubic equation with real coefficients and has a positive discriminant has three real roots and yet are expressed in terms of complex quantities. This can be shown in general from Cardan's formulas. For the case $\Delta > 0,$ A and B are shown to be complex numbers, which are conjugates of each other:

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = -\frac{q}{2} + \sqrt{-\frac{\Delta}{27}} = -\frac{q}{2} + i\sqrt{\frac{\Delta}{27}},$$

$$B = -\frac{q}{2} - i\sqrt{\frac{\Delta}{27}}.$$

Thus, the roots, which are cube roots of these expressions, are complex. It is natural to ask whether it is possible to express the roots in terms of *real radicals*. For some special cases, the answer is yes. For example, the cubic equation $x^3 + x^2 - 5x - 5 = 0$ has the roots $-1, \pm\sqrt{5}$, which are readily seen because the equation factors as $(x+1)(x^2-5) = 0.$ In most cases, the answer is no whenever the polynomial is irreducible (i.e., not factorable).

A theorem from Galois Theory: If a polynomial *f* with real roots is irreducible over a subfield $F \subseteq R$ and has degree not a power of 2, then no root of *f* is expressible by real radicals over *F*.

The significance of the cubic formula in the history of mathematics is twofold: (1) it forced mathematicians to take complex numbers and negative numbers seriously; and (2) it was the first instance of a result unknown to the ancients, showing that sixteenth-century man was the equal of his ancestors.

Remark: There is an alternate solution for the real roots of a cubic, other than Cardan's formulas, which avoids the use of complex radicals, namely the trigonometric solution known to Viete (1593). From the trigonometric identity $4cos^3\theta - 3cos\theta - \cos(3\theta) = 0,$ we can solve *any* cubic equation $y^3 + py + q = 0,$ with real coefficients and positive discriminant, which implies $p < 0,$ by a simple change of variables:

$$\theta = \frac{1}{3}cos^{-1}\left[-\frac{q}{2}\left(\frac{3}{-p}\right)^{\frac{3}{2}}\right];$$

and the roots are given by:

$$y_k = 2\sqrt{\frac{-p}{3}}cos\left(\theta + \frac{2\pi k}{3}\right), \qquad k = 0, 1, 2.$$

From Example 2, the roots of $y^3 - 3y + 1 = 0,$ are:

$$y_1 = 1.5320888862$$
$$y_2 = -1.8793852416$$
$$y_3 = 0.3472963554.$$

V. Conclusion

When we need to solve a cubic equation, there is a formula to find its roots, namely Cardan's formula. The discriminant is used to tell the nature of the roots: whether real, imaginary, or multiple roots. When the roots are all real, Cardan's formula gives the roots in terms of complex radicals. Galois theory shows that if a polynomial is irreducible and with degree not a power of 2, then its real roots are not expressible in terms of real radicals.

The roots of a cubic polynomial can always be expressed in closed form using Cardan's formula. When the roots of an irreducible polynomial are real, the formula requires taking the roots of complex quantities. Galois theory shows that this could not be helped: the real roots of an irreducible polynomial that has degree not a power of 2 cannot be expressed in terms of real radicals. The real roots may be calculated with ease using trigonometry.

References:

1. JA Beachy and WD Blair, 1996. *Abstract Algebra.* IL: Waveland Press.
2. J Bewersdorf, 2006. *Galois Theory for Beginners*. Providence, RI: American Mathematical Society.
3. D Cox, 2004. *Galois Theory*. NJ: J Wiley & Sons.
4. E Dehn, 1960. *Algebraic Equations*. NY: Dover.
5. D Dobbs and R Hanks, 1992. *A Modern Course on the Theory of Equations.* Washington, NJ: Polygonal Publishing House.
6. J Rotman, 2000. *A first Course in Abstract Algebra*. Upper Saddle River, NJ: Prentice-Hall.
7. JV Uspensky, 1992. *Theory of Equations*. NY: McGraw-Hill.