## EXPLORING THE FIBONACCI SEQUENCE OF ORDER TWO WITH CAS TECHNOLOGY

## Jay L. Schiffman, Rowan University 201Mullica Hill Road Glassboro, NJ 08028-1701 e-mail:schiffman@rowan.edu

**<u>ABSTRACT</u>**: The sequence recursively defined by  $P_1 = 1$ ,  $P_2 = 2$ , and  $P_n = 2 \cdot P_{n-1} + P_{n-2}$  for  $n \ge 3$  is often called the Pell Sequence or Fibonacci sequence of order two. In this paper, the TI-89 and MATHEMATICA 8.0 enable one to examine ratios of successive terms, explore divisibility and periodicity, furnish early prime outputs and provide algebraic proofs of palatable number tricks.

We begin by considering the sequence 1, 2, 5, 12, 29, 70, 169, 408, 985,.... This sequence is often referred to as the Pell sequence or a Fibonacci sequence of order two. Observe that the initial two terms are fixed at 1 and 2. After that, each term is the sum of twice the previous term plus the term two places before it. To cite three early examples,

 $P_{3} = 2 \cdot P_{3-1} + P_{3-2} = 2 \cdot P_{2} + P_{1} = 2 \cdot 2 + 1 = 4 + 1 = 5,$   $P_{4} = 2 \cdot P_{4-1} + P_{4-2} = 2 \cdot P_{3} + P_{2} = 2 \cdot 5 + 2 = 10 + 2 = 12 \text{ and}$  $P_{5} = 2 \cdot P_{5-1} + P_{5-2} = 2 \cdot P_{4} + P_{3} = 2 \cdot 12 + 5 = 24 + 5 = 29.$ 

The table below furnishes the initial twenty terms in the sequence:

<i>n</i> :	$P_n$ :	<i>n</i> :	$P_n$ :
1	1	11	5741
2	2	12	13860
3	5	13	33461
4	12	14	80782
5	29	15	195025
6	70	16	470832
7	169	17	1136689
8	408	18	2744210
9	985	19	6625109
10	2378	20	15994428

Examining the table in some detail for this small data set, one is led to the following conjectures concerning divisibility in this sequence:

**Conjecture 1:** Every even numbered term is even. **Conjecture 2:** Every fourth term is divisible by both 3 and 4. Conjecture 3: Every third term is divisible by 5. **Conjecture 4:** Every sixth term is divisible by 7. **Conjecture 5**: Every seventh term is divisible by 13.

The proofs of these conjectures are easily resolved via The Principle of Mathematical Induction. Let us illustrate the proof of **Conjecture 3**:

Let P(n) be the proposition  $5 | P_{3,n}$ . Step 1: Prove P(1) is true (This is the Basis Case):  $5 | P_{34} \Leftrightarrow 5 | P_3 \Leftrightarrow 5 | 5 \Leftrightarrow 5 = 5 \cdot 1$  which is true. Step 2: Assume P(k) is true (This is the Induction Hypothesis): We assume  $5 | P_{3k}$ . Step 3: Prove that  $P(k) \Rightarrow P(k+1)$ . (This establishes the induction): We prove  $5 | P_{3k} \Rightarrow 5 | P_{3(k+1)} \Leftrightarrow 5 | P_{3k} \Rightarrow 5 | P_{3,k+3}$ . Using the recursion relation in our sequence we observe that  $P_{3,k+3} = 2 \cdot P_{3,k+2} + P_{3,k+1} = 2 \cdot (2 \cdot P_{3,k+1} + P_{3,k}) + P_{3,k+1} =$  $4 \cdot P_{3\cdot k+1} + 2 \cdot P_{3\cdot k} + P_{3\cdot k+1} = 5 \cdot P_{3\cdot k+1} + 2 \cdot P_{3\cdot k}$ . By the Induction Hypothesis,  $5 \mid P_{3k} \Longrightarrow 5 \mid 2 \cdot P_{3\cdot k}$ . In addition,  $5|5 \Rightarrow 5|5 \cdot P_{3,k+1}$ . Since  $5|2 \cdot P_{3,k}$  and  $5|5 \cdot P_{3,k+1} \Rightarrow 5|[5 \cdot P_{3,k+1} + 2 \cdot P_{3,k}] \Leftrightarrow 5|P_{3,k+3} \Leftrightarrow$  $5 | P_{3(k+1)}$  establishing our induction.

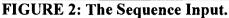
Step 4: By The Principle of Mathematical Induction, since the statement is true for n = 1, it must be true for n = 1 + 1 = 2, n = 2 + 1 = 3,... i.e.  $\forall n \in \mathbb{N}$ . This completes the proof of our contention.

The TI-89 illustrates a graph, Table, and Trace Values. Divisibility is achieved using the mod command. Consider the following screen captures in FIGURES 1-16 where the calculator is placed in SEQUENCE MODE:

ZoomEdit / All Style Axes
$\vee u1 = 2 u1(n-1) + u1(n-2)$ u11=(2 1)
u12=
ui3=
u4=
u5=
ui5=
u1(n)=2*u1(n-1)+u1(n-2)
MAIN RAD AUTO SEQ

FIGURE 1: The	Sequence Input.
---------------	-----------------

	mEdit All Style Axes	٦
PLOTS	(1(n-1) + u1(n-2))	٦
u11=10= u2= u12=		
u3= u13=		
u14= u5=		
uij=	.1>	-
MAIN	RAD AUTO SEQ	_

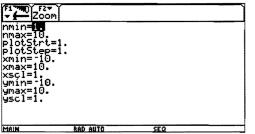


TABL	E SETUP
tblStart	1.
stbl	1.
Graph <-> Table	OFF→
Independent	AUTO÷
Enter=SAVE)	(ESC=CANCEL)

TYPE + TENTERIERK AND LESCIECANCEL FIGURE 3: The Table Setup.

7170 ▼ <b>1</b> 0 Se	F2 etup (Se) ( Heads	a Del Poelire Poel	
n	u1		
9.	985.		1
10.	2378.		1
11.	5741.		
12.	13860.		]
13.	33461.		]
14.	80782.		]
15.	195025.		]
16.	470832.		
n=9.		· · · · · · · · · · · · · · · · · · ·	
MAIN	BAD AUTO	SEQ	

FIGURE 5: The Table Revealed.



## FIGURE 7: The Standard Window.

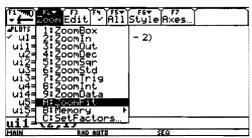


FIGURE 9: The Zoom Fit Option.

F1 TO F2 To To T	-1. -1.		
MAIN	RAD AUTO	SEQ	



s and	etup Setter	An A	antra Poor
n	ui		
1.	1.		
2. 3.	2.		
3.	5.		
4. 5. 6.	12.		
5.	29.		
6.	70.		
7.	169.		
8.	408.		
n=1.	-		
MAIN	RAD BUTS	SEC	2

FIGURE 4: The Table Revealed.

n.	lui	
17.	1136689.	
18.	2744210.	
19.	6625109.	
18. 19. 20. 21.	15994428.	
21.	38613965.	
22.	93222358.	
22.	2,25059£8	
24.	5.4334£8	

FIGURE 6: The Table Revealed.

Zoom Trace	Regraph Math Draw	P
		1
	а	
	. •	
	[	
nc:3.	uct5	
MAIN RAD A		
XCIJ. MAIN RADA	yc:5.	 is 5

FIGURE 8: The Third Term is 5.

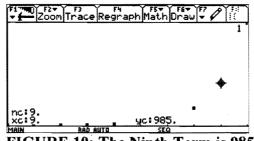
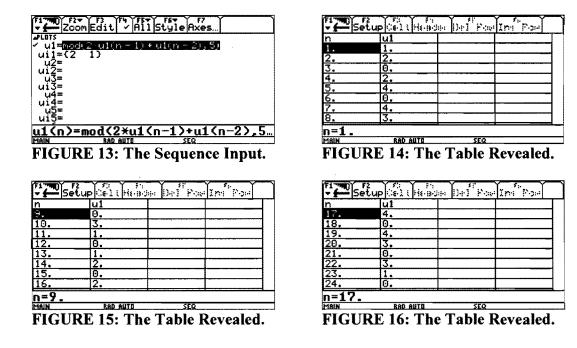


FIGURE 10: The Ninth Term is 985.

	ther PromIOClean Up
11Humber 2:Angle 3:List 4:Matrix 5:Complex 6:Statistics 7:Probability 8:Test 9:Algebra 8:Trig 8:Calculus C4Hyperbolic	1:exact( 2:abs( 3:round( 4:iPart( 5:fPart( 6:floor( 7:ceiling( 8:sign( 8:sign( 8:remain( 8:remain( B:lcm( C:gcd(
	CEO 6288

FIGURE 12: The Mod Key.



FIGURES 1-2 input the Pell Sequence, FIGURE 3 is the Table Setup, FIGURES 4-6 generate the initial two dozen terms of the sequence, FIGURE 7 is the Standard Viewing Window for the sequence, FIGURE 8 is the calculator generated graph with the third term of the sequence displayed (5), FIGURE 9 is the Zoom Fit Option, FIGURE 10 fits the first ten data points to the adjusted window depicted in FIGURE 11, FIGURE 12 illustrates the mod command, FIGURE 13 is the input to see which terms of the sequence might by divisible by five, and FIGURES 14-16 give rise to our conjecture that every third term in the sequence is divisible by five, a fact we proved by mathematical induction earlier. FIGURES 14-15 also shows the periodicity of the Pell Sequence modulo 5 (which is of length one dozen) with the sequence of remainders being 1, 2, 0, 2, 4, 0, 4, 3, 0, 3, 1, 0, 1, 2, 0, 2, 4, 0, 4, 3, 0, 3, 3, 1, 0, ....

Pr ime	Term	Output Value
2	$P_2$	2
3	<i>P</i> <sub>4</sub>	12
5	$P_3$	5
7	<i>P</i> <sub>6</sub>	70
11	<i>P</i> <sub>12</sub>	13860
13	<i>P</i> <sub>7</sub>	169
17 -	<i>P</i> <sub>8</sub>	408

Our first table illustrates the initial time the first thirty prime numbers enter the Pell sequence.

19	P <sub>20</sub>	15994428
23	P <sub>22</sub>	93222358
29	P <sub>5</sub>	29
31	P <sub>30</sub>	107578520350
37	P <sub>19</sub>	6625109
41	P <sub>10</sub>	2378
43	P <sub>44</sub>	24580185800219268
47	P <sub>46</sub>	143263821649299118
53	P <sub>27</sub>	7645370045
59	P <sub>20</sub>	15994428
61	P <sub>31</sub>	259717522849
67	P <sub>68</sub>	37774750930342781945186500
71	P <sub>70</sub>	220167382952941249990598278
73	P <sub>36</sub>	21300003689580
79	P <sub>26</sub>	3166815962
83	P <sub>84</sub>	50305164660422142002238655969020
89	P <sub>44</sub>	24580185800219268
97	P <sub>48</sub>	835002744095575440
101	P <sub>51</sub>	11749380235262596085
103	P <sub>34</sub>	3654502875938
107	P <sub>108</sub>	77308816174220163766296465781233402364740
109	P <sub>110</sub>	45058880117184178512984722732832363869422
113	P <sub>28</sub>	18457556052

Our second table furnishes a list of the values of n yielding the prime outputs in the Pell sequence  $P_n$  for  $n \le 500$ :

 $P_n$  is prime for n = 2, 3, 5, 11, 13, 29, 41, 53, 59, 89, 97, 101, 167, 181 and 191.

<i>n</i> :	Prime Output Value $P_n$ :	
2	2	
3	5	
5	29	
11	5741	
13	33461	
29	44560482149	
41	1746860020068409	

53	68480406462161287469
59	13558774610046711780701
89	4125636888562548868221559797461449
97	4760981394323203445293052612223893281
101	161733217200188571081311986634082331709
167	2964793555272799671946653940160950323792169332712780937764687561
181	677413820257085084326543915514677342490435733542987756429585398537901
191	4556285254333448771505063529048046595645004014152457191808671945330235841

The primality of *n* is necessary for  $P_n$  to be prime. The converse is not valid; for 7 is prime, but  $P_7 = 169 = 13^2$  and hence composite.

We next consider the ratios of successive terms in the Pell sequence. See **FIGURES 17-19** where one examines the ratio of an even-numbered term to the preceding odd numbered term and **FIGURES 20-22** where we consider the ratio of an odd numbered term to the preceding even numbered term.

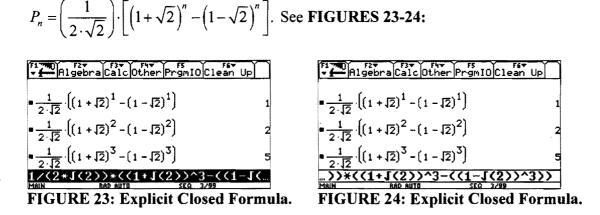
Algebra Calc Other PrgmIO Clean Up	Algebra Calc Other PrgmIO Clean Up
• 2/1 2.	■ <u>13860</u> 5741 2.41421355165
• 12/5 2.4 • 70/29 2.41379310345	<b>80782</b> <b>33461</b> 2.41421356206
• 408 • 169 2.41420118343	470832 ■ <u>195025</u> 2.41421356236
2378 985 2.41421319797	2744210 1136689 2.41421356237
2378/985	2744210/1136689
MAIN RAD AUTO SEQ 5/99	MAIN RAD AUTO SEC 9799
FIGURE 17: Ratios of Successive Terms.	FIGURE 18: Ratios of Successive Terms.
F17700 F2▼ F3▼ F4▼ F5 ▼ ← Algebra Calc Other PrgmIO Clean Up	F1 mm → ← Algebra Calc Other PrgmIO Clean Up
▼ = nigebralcaic uther rrghi0 ciean op	■ 5/2 2.5
	■ 29/12 2.41666666667
	$=\frac{169}{70}$ 2.41428571429
	■ <u>985</u> 408 2.41421568627
• <u>15994428</u> • <u>6625109</u> 2.41421356237	• <u>5741</u> 2.41421362489
15994428/6625109	5741/2378
MAIN RAD AUTO SEO 1/99	MAIN BAD AUTO SEQ 5/99
FIGURE 19: Ratios of Successive Terms.	FIGURE 20: Ratios of Successive Terms.
(F1 790) F2 7 F3 F F4 F5 F5 F6 F6 F	(F1 790) F2 1 F3 F F F4 F5 F5 F6 F6 F
+ Algebra Calc Other PrgmIO Clean Up	- Algebra Calc Other PrgmIO Clean Up
■ <u>33461</u> <u>13860</u> 2,41421356421	
■ <u>195025</u> 80782 2.41421356243	
• <u>1136689</u> <u>470832</u> 2.41421356237	
6625109	70617065

6625109/2744210MainSEQ9/99FIGURE 21: Ratios of Successive Terms.FIGURE 22: Ratios of Successive Terms.

2.41421356237

2,41421356237

The sequence in **FIGURES 17-19** is clearly increasing while the sequence in **FIGURES 20-22** is clearly decreasing. Both sequences are approaching the same number; namely  $1+\sqrt{2}$ . This is the Golden Mean analogue of the number  $\Phi = \frac{1+\sqrt{5}}{2} \approx 1.61803398875$  in the Fibonacci sequence. This Pell sequence constant satisfies the quadratic equation  $x^2 = 2 \cdot x + 1$  in much the same manner that the constant  $\Phi = \frac{1+\sqrt{5}}{2}$  satisfies the quadratic equation  $x^2 = x + 1$  in dealing with the Fibonacci sequence. While the n-th Fibonacci number  $F_n$  satisfies the Binet Formula  $F_n = \left(\frac{1}{\sqrt{5}}\right) \cdot \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ , the n-th Pell number  $P_n$  satisfies the Binet-like formula



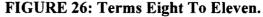
Our concluding activities focus on some interesting number tricks associated with this sequence as well as securing the prime factorizations of the first twenty-five Pell numbers.

Consider the sum of any four, eight, and twelve consecutive terms in the Pell sequence. Let the first two terms be x and y respectively. The next terms can be secured by the TI-89 as in **FIGURES 25-27**:

Algebra Calc Other PrgmI	Clean Up
• ×	×
	. y
= 2·y+x	x+2·4
$= 2 \cdot (x + 2 \cdot y) + y$	2·x+5·y
■2·(2·x+5·y)+x+2·y	5·x+12·y
= 2·(5·x + 12·y) + 2·x + 5·y	12·×+29·y
= 2·(12·x + 29·y) + 5·x + 12·y	29·x + 70·y
2*ans(1)+ans(2)	2/99
FININ BRU AUTU SEU	((99

FIGURE 25: The First Seven Terms.

A1	f2+ F3+ F4+ gebra Calc Other	PrgmIOClean Up
2.(29.)	x + 70'y) + 12·x	+ 29 · y 70 · x + 169 · y
■ 2·(70·)	k + 169 ·y) + 29 ·×	(+70·y 169·x+408·y
• 2·(169	•x +.408 •y) + 70 •	x+169·y
■ 2·(408	•x + 985 •y) + 169	408 · x + 985 · y 9 · x + 408 · y
2×ans	(1)+ans(2)	985 · x + 2378 · y
MAIN		SEQ 11/99



F1 790) F2 Y F3 Y F4 Y F5 Y F6 Y
FirmO Fix
169·× + 408·y
= 2·(169·x + 408·y) + 70·x + 169·y
408 · x + 985 · y
= 2·(408·x + 985·y) + 169·x + 408·y
985 x + 2378 y
= 2·(985·x + 2378·y) + 408·x + 985·y
2378 x + 5741 y
2*ans(1)+ans(2)
MAIN RAD AUTO SEQ 12/99
FIGURE 27: The Twelfth Term.

Note that the coefficients of the terms are all Pell numbers. Now the sum of four consecutive terms is

 $x + y + x + 2 \cdot y + 2 \cdot x + 5 \cdot y = 4 \cdot x + 8 \cdot y$ . Note that  $\frac{4 \cdot x + 8 \cdot y}{4} = x + 2 \cdot y$ , the third term in the sequence.

The sum of four consecutive terms in the Pell sequence is divisible by four and the quotient is the third term in the sequence. See FIGURE 28 where the calculator furnishes a proof:

F1770 F27 F37 F37 F47 F5 F1770 Algebra Calc Other PrgmI	0 Clean Up
■x+y+x+2·y+2·x+5·y	4∙x+8∙y
$\frac{4 \cdot x + 8 \cdot y}{4}$	x + 2 · y
ans(1)/4	
MAIN RAD AUTO SEQ	2/99

FIGURE 28: A Calculator Proof.

Next examine the sum of eight consecutive terms in the sequence and divide the sum by twenty-four. The quotient is the fifth term in the sequence. See FIGURES 29-30:

F1 Algebra Calc Othe	rPrgmIOClean Up	F1700 Ageira Calc Other Pi
■ x + y + x + 2 · y + 2 · x + 5	j·y+5·x+12·y+12≯ 120·x+288·y	■ <b>(</b> y+12·x+29·y+29·x+7
= <u>120·×+288·y</u> 24	5·x + 12·y	$=\frac{120 \cdot x + 288 \cdot y}{24}$
(120*x+288*y)/24	SEQ 2/99	(120*x+288*y)/24
FIGURE 29: A Ca	FIGURE 30: A Calcu	

Pigebra Calc Other	PromIO Classifier
= <b>∢</b> µ + 12 · × + 29 · µ + 29 · ×	and the second sec
	120 · x + 288 ·
<u>120·×+288·y</u>	5·x + 12·
24	
(120*x+288*y)/24	SEQ 2/2
MAIN RAD AUTO	
FIGURE 30: A Cal	culator Proof.

Finally the sum of twelve consecutive terms is divisible by 140 and the quotient is the seventh term in the sequence. See FIGURES 31-34:

F1→m0 → ∰ Algebra Calc Other	rs PrgmIO[Clean Up]
■ x + y + x + 2 · y + 2 · x + 5 · y	y + 5·x + 12·y + 12 ) 4060·x + 9800·y
■ <u>4060·× + 9800·y</u> 140	29∙x + 70∙y
	SEQ 2/99
FIGURE 31: A Calc	ulator Proof.

	PrgmIOCLARY Up
■ <b>《</b> +169·×+408·y+408	
<u>4060 · x + 9800 · y</u> 140	29 · x + 70 · y
ans(1)/140	SEQ. 2/2
FIGURE 33: A Cal	lculator Proof.

FLTON, ST LITY	Sec PromIO Clean Up
The presence of the	ater PromIO Classe Up
■ 📢 12·× + 29·g + 29·s	
	4060 · x + 9800 ·
_ 4060·x + 9800·y	
140	29·×+70·
ans(1)/140	
MAIN RAD AUTO	SEQ 2/2
FIGURE 32: A C	
	Calculator Proof.
Platera Calcos	Calculator Proof.
	Calculator Proof.
Platera Calcos	Calculator Proof.
Platera Calcos	Calculator Proof.
Interaction Inter	Calculator Proof.
[1]	Calculator Proof.

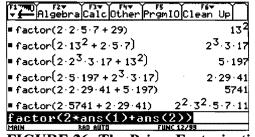
FIGURE 34: A Calculator Proof.

We conclude by securing the factorizations of the first twenty-five Pell numbers in **FIGURES 35-49** below:

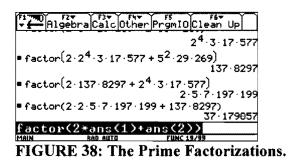
Algebra Calc Other PrgmIOClea	an Up
= 1	1
= 2	2
<pre>factor(2·2 + 1)</pre>	5
• factor(2·5 + 2)	2 <sup>2</sup> .3
• factor $(2 \cdot 2^2 \cdot 3 + 5)$	29
• factor $(2 \cdot 29 + 2^2 \cdot 3)$	2.5.7
factor(2*ans(1)+ans(2))	. –
MAIN RAD AUTO FUNC 6/99	

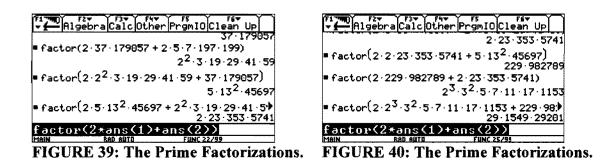
FIGURE 35: The Prime Factorizations.

Algebra Calc Other PrgmIO Clean Up
• factor $(2 \cdot 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 + 5741)$ 33461
= factor $(2 \cdot 33461 + 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11)$
2 · 13 <sup>2</sup> · 239
= factor $(2 \cdot 2 \cdot 13^2 \cdot 239 + 33461)$ 5 <sup>2</sup> · 29 · 269
= factor $(2 \cdot 5^2 \cdot 29 \cdot 269 + 2 \cdot 13^2 \cdot 239)$
24.3.17.577
factor(2*ans(1)+ans(2))
MAIN RAD AUTO FUNC 16/99
FIGURE 37: The Prime Factorizations.









**Conclusion:** Fibonacci-like sequences are very amenable for discovering neat insights that have broadly based appeal. This paper explored some elementary ideas with the Fibonacci sequence of order two also referred to as the Pell sequence in the literature. Applications of this sequence abound. One may extend the ideas extrapolated in this paper by forming new conjectures associated with this and other Fibonacci-like sequences. For example, Neil J.A. Sloane, the founder of the excellent OEIS (The On-Line Encyclopedia of Integer Sequences) presents the initial five hundred terms in this sequence. I have utilized MATHEMATICA 8.0 to secure the prime factorizations of all but one of the initial three hundred members of this sequence; namely the two hundred ninety-ninth which is currently in progress. A very neat application to geometry is manifested in the Pell sequence with regards to right triangles commonly known as Theon's Ladder which is the subject of a stimulating article in the *College Mathematics Journal* co-authored by a colleague of mine with one of his top students.

## **Reference:**

1. Thomas J. Osler (with Shaun Giberson), *Extending Theon's Ladder to any Square Root*, The College Mathematics Journal, 35 (2004), pp. 222-226.