Discrete Dynamical Systems with the TI-Nspire CAS Part II

Dr. Rich West
Francis Marion University
PO Box 100547
Florence, South Carolina 29502
rwest@fmarion.edu

Over the past twenty years I have taught Discrete Dynamical Systems (DDS) at many different levels, from high school to graduate school. Initially I started teaching freshmen at West Point, and my latest endeavors have been again at the freshmen level at Francis Marion. In the past my technology of choice has been the TI-84 or TI-83 Plus, but the last two years I have tried working with the TI-Nspire CAS. In last year's presentation (Part I) I spent most of my time talking about the technology and very little time talking about dynamical systems. Thus my article (see reference) tried to fill in the gaps with both specific technology steps used with specific dynamical systems. This year's article will be more expository in addressing my favorite project in dynamical systems, the embezzler problem. Further, I will compare and contrast the TI-84/83 and the TI-Nspire CAS in addressing this specific problem.

First, here is a quick review for those not familiar with discrete dynamical systems. DDSs are powerful, intuitive modeling tools for describing and predicting changing situations. Using a DDS involves modeling with the paradigm:

future = present + change.

A DDS is a discrete function that can be used to model many situations, such as mortgage of a home, car financing, investment or financial alternatives, prescribed drug dosages, population dynamics such as predator-prey problems, competitive hunter problems, genetics, and pretty much anything involving change. Further, the discrete nature of these models allows us to use technology to seek answers where answers are many times unavailable in the continuous analog, differential equations. Discrete dynamical system models are some of the more robust models known. A DDS might sound familiar from calculus, the study of change. In calculus our paradigm is changed to $change = future\ value - present\ value$ which is known as a difference equation.

The Embezzler Problem. Here's the Embezzler Problem as presented to my freshmen class this past year:

When you are 36, you manage to land a fantastic job, and you start a savings account with 20,000 dollars invested at 10 percent interest compounded annually. Next year, you will deposit an additional 10,000 dollars, and then each year thereafter 20 percent more than the year before. You intend to retire in 24 years. Unknown to you, there is an evil embezzler at the bank who decides to steal 2000 dollars from your account immediately, and then each year thereafter, 30 percent more than the year before. The embezzler is a master at fixing the books so that no one suspects anything. The very day that you retire and want your money the embezzler safely skips country. Alas, the bank claims their

records show that you made withdrawals yourself, and a nightmare lawsuit ensues. Eventually, you want answers to the following questions:

- How much money did you expect to have for your retirement?
- How much money did the villain take out of the country?
- How much money did the villain considerately leave in your account to pay for your lawsuit?

We will approach each question separately.

How much should you have on retirement? As good modelers, we start by defining our variables. $r(n) =$ the number of dollars that should be in the retirement plan after n years. Next we either use a word picture or a diagram to show our model of the change that takes place each year. To simplify things I model the deposit separately. So I need some more variables. $d(n) =$ the number of dollars in the deposit made in the nth year. To model the deposit, we use "future = present + change" to get the word picture "new deposit = old deposit + 20% of the old deposit." This yields the discrete dynamical system $d(n) = d(n-1) + 0.2d(n-1)$ with the initial condition $d(0) = 10000$. Now we can write the word picture and model for our retirement plan. "This year's plan = last year's plan + 10% of last year's plan + the deposit." Or $r(n) = r(n-1) + 0.1r(n-1) + d(n-1)$ with the initial condition $r(0) = 20000$. We arrived at this equation after writing out the first couple years: We had an initial deposit of $20000. Thus $r(0) = 20000$. Then we earn 10% interest of this $20000 and we deposit $10000 the first year. Thus $r(1) = 20000 + 0.1*20000 + 10000 = 32000$. For year two we have $32000 plus the 10% interest accrued on the $32000 plus our new deposit, which for the second year is $10000 plus 20 % of $10000 or $12000. Thus $r(2) = 32000 + 0.1*32000 + 12000 = 47200$. Now let's check out our models

$$d(n) = d(n-1) + 0.2d(n-1), \quad d(0) = 10000$$

$$r(n) = r(n-1) + .1r(n-1) + d(n-1), \quad r(0) = 20000$$

Using these models $d(0) = 10000$ and $d(1) = 12000$, while $r(0) = 20000$, $r(1) = r(0) + 0.1r(0) + d(0) = 20000 + 2000 + 10000 = 32000$,

and $r(2) = r(1) + 0.1r(1) + d(1) = 32000 + 3200 + 12000 = 47200$. So our model works!

Now let's use the TI-Nspire to answer our question. Go to the Spreadsheet. If you want a graph, then I recommend that you name your variables. To do this move the cursor up two spaces and type in *year* and ENTER. Move the cursor to the right one space and up and type in *deposit* and ENTER. Move to the right one space and up and type in *plan* and ENTER. Now use the cursor to move to cell A1. Type in 0 and ENTER. In cell A2 type in *=a1*+1 and ENTER. Move back up to cell A2 and press the hand (in the center of the cursor) and hold the button down until the dashed outline appears. Then press the down cursor until you get to cell A25 (the dashed outline should travel with you and outline the column) and then press ENTER. You should now have 0 through 24 in column A. Next, move to the right and up one space to cell B1. Here we put our first deposit of $10000. So type in 10000 and press ENTER. In cell B2 type in *=b1+.2*b1* and ENTER. Now move back up to cell B2 and press and hold the hand until the dashed outline appears. Then press the down cursor to cell B25 (next to the 24 in the year column) and press

ENTER. You should now have a column of all the deposits for years 1 through 25. Next move one space to the right and up to cell C1. Here we type in 20000 for our initial deposit into the retirement plan. In cell C2 type in $=c1+0.1*c1+b1$ and ENTER. 32000 should appear in this cell. Move to cell C2 and press and hold the hand until the dashed outline appears. Move the cursor down to C25 and press ENTER. In cell C25 is the answer to our question: our retirement plan should have made $7,161,706.11. If it does not, check the bottom line which should read $C25=c24+0.1c24+b24$. This is the formula for doing the last iteration. Also check cell C3 which should read $47,200, our answer for $r(2)$.

How much did the embezzler take out of the country? Now that we know we should have over 7 million dollars in our retirement fund, we can focus on the embezzler. Mostly we are interested in what the embezzler took from us. Let's define some new variables: $e(n)$ = the number of dollars taken in the nth year, and $t(n)$ = the number of dollars the embezzler has accumulated after n years. A word picture for "the amount taken in a year = the amount taken last year + 30% of the amount taken last year." So in DDS terms $e(n) = e(n-1) + .3e(n-1)$ with the initial condition $e(0) = 2000$. As far as answering the question, all we need to do is sum up all of the takeouts. So in a word picture "the amount that the embezzler accumulated this year = the amount accumulated last year + the amount taken this year." In DDS terms $t(n) = t(n-1) + e(n)$ with the initial condition $t(0) = 2000$.

A quick mental check says that $t(0) = 2000, t(1) = 4600,$ and $t(2) = 7980$. Using our DDSs $e(0) = 2000, e(1) = 2600,$ and $e(2) = 3380$ and $t(0) = 2000, t(1) = t(0) + e(1) = 2000 + 2600 = 4600,$ and $t(2) = t(1) + e(2) = 4600 + 3380 = 7980$. So our model works!

Next we are on to the TI-Nspire to answer our question. We start in a new spreadsheet and create our titles at the top of the columns. I called mine *year*, *ntake*, and *taken*. Create a column of years from 0 to 24 just like we did for the previous question. Then go to cell B1 and type in 2000 and ENTER. In B2 we type in "$=b1+0.3*b1$"and ENTER. Then go back up to B2, hold down the hand until the dashed outline appears, scroll down to B25, and hit ENTER. As a check, the takeout in the 24th year is $1,085,600. Next go to cell C1 and type in 2000. In C2 type "$=c1+b2$" and ENTER. Go back to C2, hold down the hand until the dashed outline appears, scroll down to C25, and hit ENTER. The answer to our question is $t(24) = \$4,697,606.68$. Wow!

How much money is really left in the account? For this model we need our deposit function and our embezzler takeout function. So we need the following variables: $d(n)$ = the number of dollars deposited in the nth year, $e(n)$ = the number of dollars taken in the nth year, and $rf(n)$ = the number of dollars in the retirement plan final after n years. The DDSs for $d(n)$ and $e(n)$ are the same as in the previous two problems. The word picture $rf(n)$ is "the amount in the retirement plan final this year = the amount in retirement plan final last year + 10% of last year's retirement plan final + the amount of last year's deposit − what the embezzler took out this year." The initial condition will be

somewhat different because the embezzler took out $2000 immediately. So $rf(0) = 18000$. This results in the following system of DDSs:

$$d(n) = d(n-1) + 0.2d(n-1), \quad d(0) = 10000$$

$$e(n) = e(n-1) + 0.3e(n-1), \quad e(0) = 2000$$

$$rf(n) = rf(n-1) + 0.1rf(n-1) + d(n-1) - e(n), \quad rf(0) = 18000$$

To check out our model, we know that $d(n)$ and $e(n)$ work and that $rf(0) = 18000$. So let's try to check $rf(1)$. Starting with $18000, we add 10% of 18000, or 1800, leaving 19800. Then we add the $10,000 deposit, leaving $29,800. Finally the embezzler takes out $2600, leaving $27,200.

So $rf(1) = rf(0) + .1 * rf(0) + d(0) - e(1) = 18000 + 1800 + 10000 - 2600 = 27200$. It check's!

Let's go to the TI-Nspire to answer our question. In a new spreadsheet we go up to the top row and put in the titles *year*, *deposit*, *ntake*, and *final*. In column A we put in 0 through 24, just as we did in the two previous problems. In column B, we put in the *deposit* just as we did for the first question. To check, $d(24) = 794,968.47$. In column C, we put in what the embezzler took out as we did in the second question. To check, $e(24) = \$1,085,600$. Next go to cell D1 and type in 18000 and ENTER. In cell D2 type"=*d1*+0.1**d1*+*b1*-*c2*" and ENTER. This should be our check value of 27200. Go back up to D2, hold down the hand until the dashed outline shows, then scroll to d25, and hit ENTER. The final answer is that after 24 years of saving and being embezzled we have a paltry $213,643 left compared to our planned $7 million.

Comparing the TI-Nspire CAS with the TI-84/83. Initially, in taking on the TI-Nspire I felt like the learning curve was too steep. But now that I've written this article, I'm not so sure. To do DDSs on the TI-84/83 you need to become very familiar with the sequence mode of the calculator. Most students I have encountered are not familiar with the sequence mode of the calculator. So from a student point of view either calculator has a steep learning curve.

From a graphing point of view and where I am on my TI-Nspire learning curve, the TI-84/83 wins hands down. But many problems do not need a graphing approach such as the one we just completed. Further, I have seen many good TI-Nspire instructors do whatever they wanted with the graphs. So while the TI-84/83 has many nice graphing formats (of which I am familiar) such as time graphs, cobwebs, and phase portraits, I'm sure that the TI-Nspire can do these as well.

At this point in my learning the TI-Nspire, I think that either technology has its advantages. Now let address this particular problem and the two technologies.

The TI-83/84 has only three sequences available: $u(n)$, $v(n)$, and $w(n)$. Therefore, these three questions need to be addressed as three separate problems. Plus on the TI-84/83 we would have to keep the different u's straight between the different problems. The TI-Nspire has plenty of columns and could handle this entire project with one spreadsheet.

Further each of the variables could have names and could be carried back and forth to the different modes of the calculator, such as graphs.

One other issue with the TI-84/83 occurred when addressing the third question. Our final equation utilized *final* as a function of $e(n)$ instead of $e(n-1)$. Because the TI-Nspire utilizes a spreadsheet this was easily handled by just utilizing the correct table value. But this is not allowed on the TI-84/83. All sequences must be functions of previous values $v(n-1)$ instead of $v(n)$. It could be worked around on the TI-84/83, but it required some ingenuity that my freshmen had difficulty with.

As time goes on in teaching my freshmen course, I will probably use the technology that is most readily available. But I can see many advantages to the TI-Nspire, and I know that as this technology becomes more readily available I will be quick to make the big switch. Further, for my higher level courses where I have calculators enough to use I will continue to explore the discrete functions of TI-Nspire CAS.

References

West, R. D. (2010). Discrete Dynamical Systems with the TI-Nspire CAS. Proceedings of the Twenty-First Annual International Conference on Technology in Collegiate Mathematics (pp.295-299) Reading, MA: Addison-Wesley.