DYNAMICAL MATHEMATICS WITH *MATHEMATICA*

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Abstract

The teaching of mathematics has undergone fundamental changes in favor of multiple representations of concepts. Particularly, there is now a greater emphasis on the visual aspect of mathematics due to the advancement of technology. The dynamical feature of many of the mathematical software programs available for the learner can largely contribute to an enhancement of the understanding of mathematical concepts even for those students who are not visualizers. *Mathematica* has recently introduced such a feature through the Manipulate command. Examples of how *Mathematica* is exploited in undergraduate classes are presented. In one example, polar curves such as multi-leaf roses are drawn dynamically for various values of n. In another example, the phase portraits for linear systems of differential equations $\frac{dx}{dt} = a_1 x + b_1 y; \frac{dy}{dt} = a_2 x + b_2 y$ are explored dynamically for various values of a_1, b_1, a_2, and b_2. In the last example, the fixed points of the quadratic family $f(x) = x^2 + c$ are investigated for various values of *c*; in the various cases, the fixed points are classified as attracting, repelling or neutral, and the bifurcation concept is introduced.

Introduction

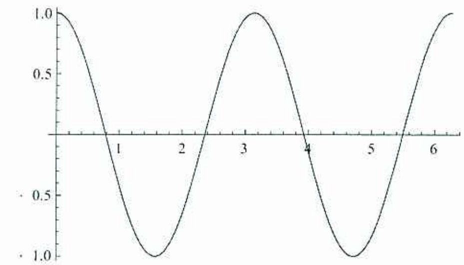
At a time marked with tremendous advances of educational technological aids, such as CAS and specialized dynamical software programs, the role of the teacher is becoming more and more critical as the primary user of technology in the classroom. For many researchers, technology provides a setting for conjecture and creativity for students and teachers alike. However, and according to Monaghan (2004), understanding the actual situation of teachers who are using technology is a complex issue. In fact, even though the young generation of teachers seem to have a positive perception of technology (Abboud-Blanchard, 2005), yet "teaching with digital tools does not simply mean considering the software and hardware used" (Monaghan, p. 339). When using technological aids, teachers have to face two main issues: deciding on the software programs to use and most importantly the design of student tasks (Laborde, 2001). According to Blyth and Labovic (2009), the "development of innovative teaching materials, which exploit the strengths of the CAS for tightly integrated eLearning and eAssesment …, can be used to produce active and engaged learners of mathematics and the technology".

Dynamical software programs are more and more common and contribute largely to a visual understanding of mathematical concepts. The mathematical education community is now emphasizing the role of multiple representations of concepts, of which visualization is one. According to the National Council of Teachers of Mathematics standards (NCTM, 2000), multiple representations are a tool that helps students solve problems, support their understanding of concepts, and communicate mathematical ideas. In a dynamic environment, the benefits of a visual representation are doubled, to say the least. According to Noss and Hoyles (1996, p. 245), when the algebraic representation is complemented with the dynamic environment, it "comes alive". In addition, such an environment "constitutes a rich experimental arena: Students receive the feedback as a direct consequence of their actions and not as a judgmental statement from their teachers" (Arcavi, A. 2008).

This paper presents the teacher's attempts at the Lebanese American University in Beirut, Lebanon, to exploit a new dynamical feature of the computer algebra system *Mathematica* (the Manipulate command) to introduce and clarify concepts in a 3rd semester calculus course, in an introductory differential equations course, and in a dynamical systems course. Students enrolled in these classes come from varied educational backgrounds and their major fields of study vary between engineering, math education, and computer science.

Graphing in Polar Coordinates

In a third semester calculus course, the topic of polar coordinates is discussed in somehow great details. In particular, a considerable amount of time is spent graphing polar curves. This is a topic that constantly shows to be a challenge for the students because graphing is done in the Cartesian plane, yet points are plotted based on their polar coordinates. Two strategies are introduced to the learners: One is straightforward, yet not very accurate, and suggests that the sketch be done based on a list of points; the other is more accurate and proposes sketching the corresponding Cartesian curve and then adapting the information into the language of polar coordinates. Thus a change in x (which is a horizontal motion) corresponds to a change in θ (which is a rotational motion), while a change in y (vertical motion) corresponds to a change in r (a motion that depends on θ). Even though the polar curve is now accurate, the exercise by itself is challenging. Figure 1 below is for the curve $r = \cos(2\theta)$ and its graph, a 4 - leaf rose, represents one of the more difficult yet interesting examples. The plot is challenging because the r values can be negative and thus the plot jumps from the first quadrant, to the third, to the fourth, to the second, back to the third, then to the first, to the second, and finally to the fourth. The example is also interesting because the leaves are not drawn/completed one at a time (see Figure 2 for snapshots of the plot). For this reason, seeing the graph in a dynamical environment can be useful for a better understanding of the curve and can generate enthusiasm among students.



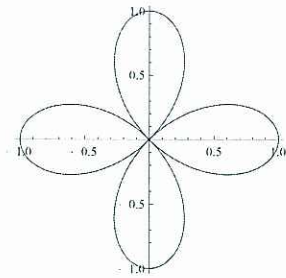


Figure 1: Cartesian vs. polar sketch of the curve $r = \cos(2\theta)$

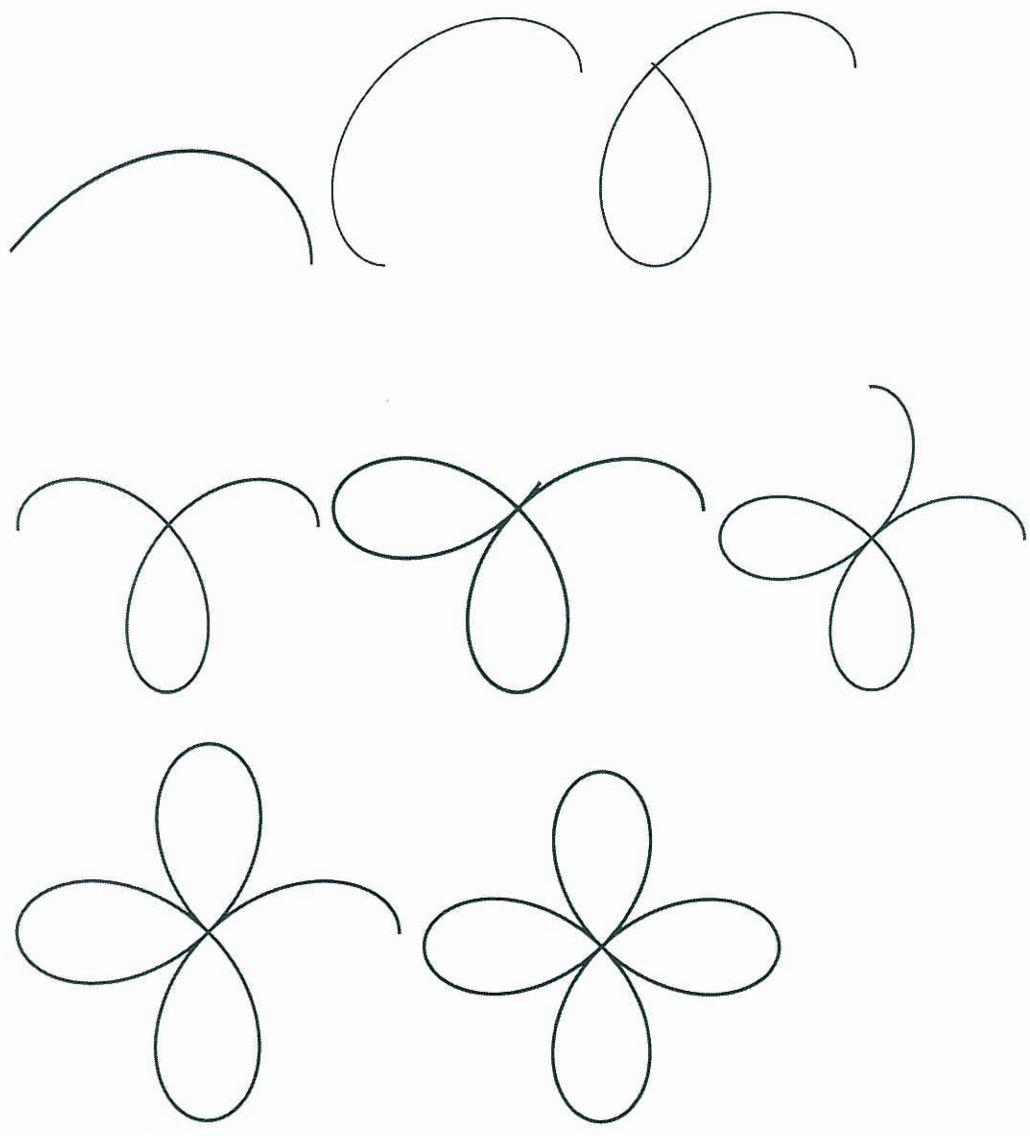


Figure 2: Snapshots of the polar curve $r = \cos(2\theta)$ as generated by *Mathematica*

As a teacher, one can exploit this interest and expand the discussion to issues that require critical thinking. For instance, the graphs below (Figure 3) done in *Mathematica* allows the leaner to change functions and change their arguments by dynamically varying n ($r = \cos(2\theta)$ vs. $r = \sin(2\theta)$). Questions students can explore are: 1. How is the number of leaves related to n? 2. What distinguishes the rose generated by a cosine function from the one generated by the sine function?

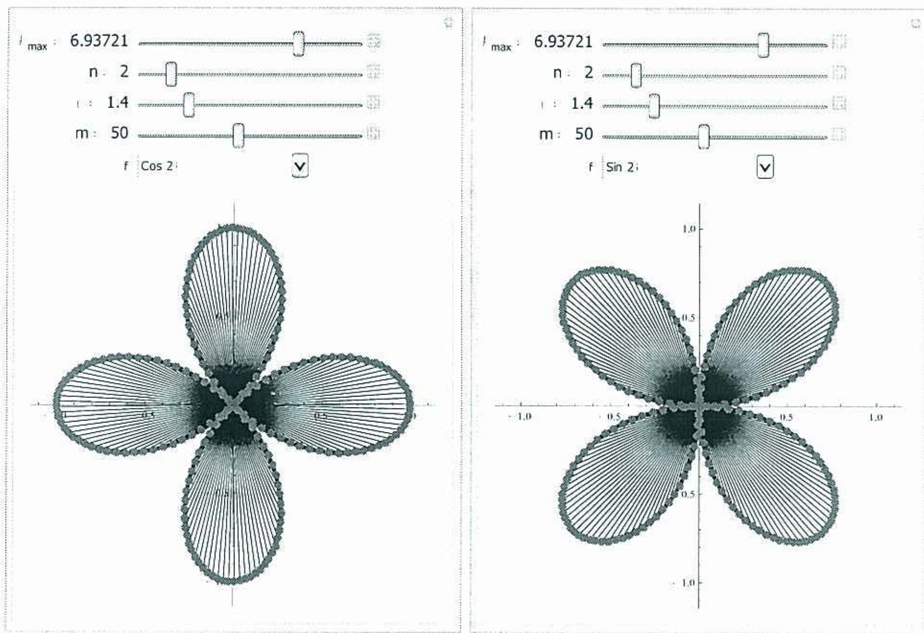


Figure 3: the graphs of $r = \cos(2\theta)$ and $r = \sin(2\theta)$ as generated by a *Mathematica* program

Phase Portraits for Linear Systems

A non-traditional introductory course in differential equations emphasizes a qualitative approach to the learning of the material. The recipe-type approach for solving quantitatively first-order differential equations is marginalized; instead such equations are solved by sketching their solutions either by means of a direction field or through a sketch of the phase line for autonomous equations. Linear systems of differential equations are also an important topic studied in such an introductory course. In the non-traditional setting, an emphasis is placed on visualizing solutions and analyzing them rather than solving equations algebraically and trying to find the solutions in a closed form format. As Hubbard puts it (1994, p. 372), "the search for formulas often obscures the central question: How do solutions behave?" In the reformed setting, second-order differential equations $\frac{d^2y}{dt^2} = f(t, y)$ are discussed after transforming them into linear systems of first-order differential equations. This is done by introducing the variable $\frac{dy}{dt} = v(t)$ and rather than solve for $y(t)$, one can view the solution curve in the *yv*-plane. Thus an understanding of the system solution curves contributes to an understanding of the equation solutions. In particular, the learner is able to investigate the long term behavior of the solutions, something almost inconceivable in the traditional approach. While such an approach in the past was not popular because of the difficulties associated with the visual aspect, the advancement of computer graphics particularly the development of specialized dynamical software programs has provided teachers and students alike with unprecedented visual means to explore dynamically second-order differential equations.

In general, linear systems of differential equations take the form $\frac{dx}{dt} = a_1x + b_1y; \frac{dy}{dt} = a_2x + b_2y$. A complete understanding of such systems requires an analysis of the coefficient matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$. More precisely, one has to analyze the characteristic equation of the matrix $\lambda^2 - 2tr(A)\lambda + \det(A) = 0$, where $tr(A)$ and $det(A)$ are the trace and determinant of A respectively. The shape of the solution curve in the *yv*-plane depends on the type and number of the eigenvalues and also on the corresponding eigenvectors. In *Mathematica,* the <u>manipulate</u> feature allows the learner to view the changes in the phase portrait in a dynamical fashion. Below (Figure 4) is an example that is available online through the *Mathematica* website. A manipulation of the matrix coefficients a_1, b_1, a_2, and b_2 produces a dynamic environment allowing the student to see in real time how for instance straight-line solutions can collapse to become one, and how the one straight-line solution disappears in favor of solution curves that spiral around the origin (the equilibrium solution). In such systems, one is interested in viewing the <u>time-series</u> of the solutions, that is the corresponding *ty* and *tv*- sketches since the *ty* time series are the actual solutions of the differential equation. Instructors can exploit the dynamical nature of the software and request from students an analysis of the different time series (shapes and long-term behavior).

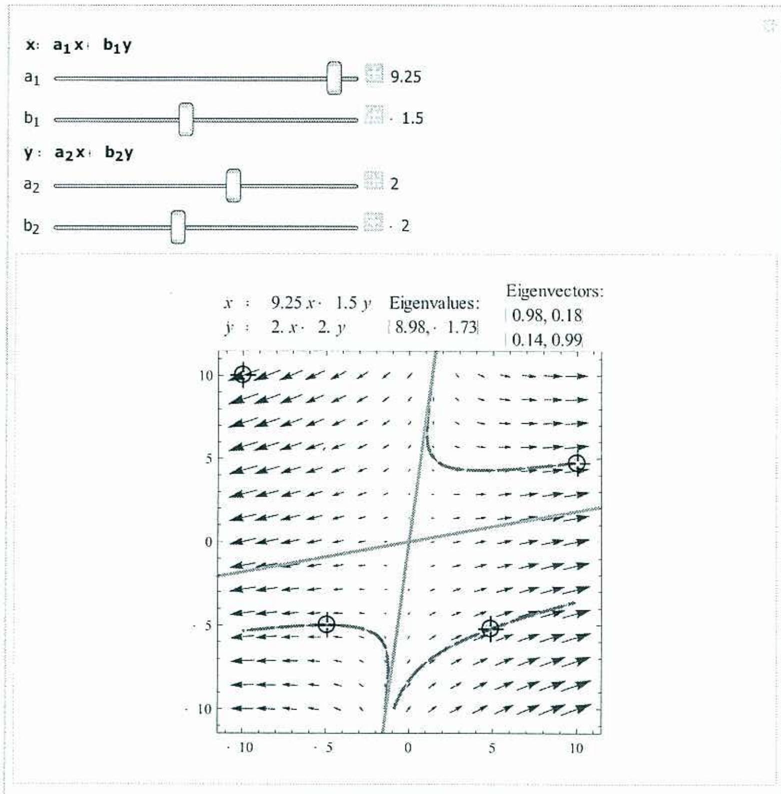


Figure 4: A dynamical phase portrait for a linear system generated by a *Mathematica* program

The Quadratic Family $f(x) = x^2 + c$

A course in dynamical systems is becoming more and more prevalent in any undergraduate mathematics curriculum. According to Devaney (1992, p. v), "the study of dynamics offers mathematicians an opportunity to expose students to contemporary ideas in mathematical research". He adds to say that many of the ideas in such a course have been discovered within the students' life span. Being a course in dynamical systems, dynamical computer programs are invaluable aids. In the example below, the Manipulate feature of *Mathematica* is used to investigate the dynamics of the quadratic map $f(x) = x^2 + c$, where c is a parameter. In such a problem, the task is to find the fixed points ($f(x) = x$) and then to qualify them as attracting, repelling, or neutral. The existence, the number, and the classification of the fixed points depend on c. A bifurcation occurs when the number of fixed points changes or when the nature of the fixed point changes. It is therefore imperative that a dynamical tool be used to complement the analytic discussion of the quadratic family. One can ask the students to explore the following questions: 1. What happens when $c > \frac{1}{4}$? 2. What if $c = \frac{1}{4}$? 3. What happens if $c < \frac{1}{4}$? 4. In the latter case, there is another bifurcation occurring when

$c = -\frac{3}{4}$. Investigate that bifurcation. Figure 5 below is a *Mathematica* program that allows the student to explore the problem in a dynamical fashion.

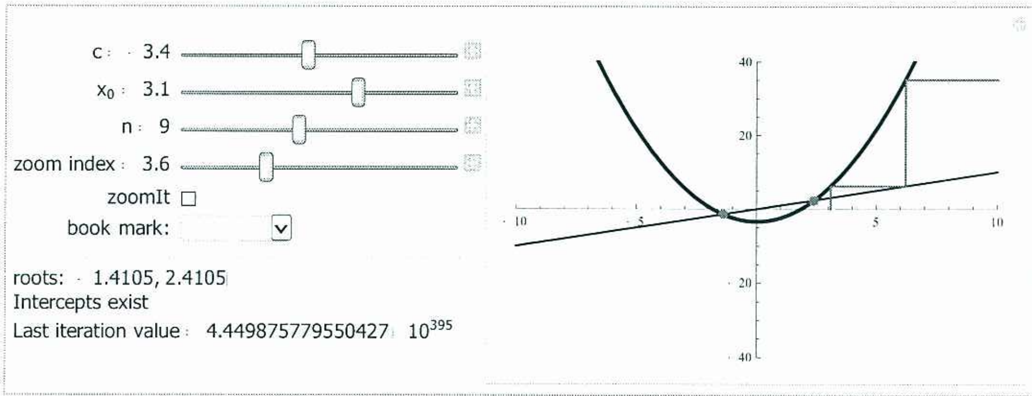


Figure 5. A dynamical investigation of the quadratic family as generated by *Mathematica*

Conclusions

In conclusion, there are many instances where the dynamical feature of *Mathematica* can be used to improve the learning environment in a classroom. Teachers are encouraged to explore the software strengths. Many have already done so and thousands of demos can be found on the Wolfram Demonstration Projects site: http://demonstrations.wolfram.com/?buildid=1213965.

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