Game Theory and Optimization via Excel

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Introduction

In this brief article, we highlight the use of linear and nonlinear programming in finding the solutions via Nash equilibriums and Nash Arbitration to game theory problems. In 1951 linear programming was shown as a viable method for solving mixed strategy zero-sum games. We begin with a variation of the formulation presented in 1951 by Dorfman for zero-sum games. We present our approach in equation (1).

Maximize V (1)

Subject to:

$$M_{1,1}x_1 + M_{1,2}x_2 + \ldots + M_{1,n}x_n - V \geq 0$$
$$M_{2,1}x_1 + M_{2,2}x_2 + \ldots + M_{2,n}x_n - V \geq 0$$
$$\vdots$$
$$M_{m,1}x_1 + M_{m,2}x_2 + \ldots + M_{m,n}x_n - V \geq 0$$
$$x_1 + x_2 + \ldots + x_n = 1$$
$$V, x_i \geq 0$$

Zero Sum Games

Example 1. Consider the game theory problem for the hitter-pitcher duel. We will consider a duel between Mickey Mantle and Warren Spahn.

LP Example EXCEL

Hitter-Pitcher Duel

		Spahn	
		FB	CB
Mantle	FB	0.45	0.23
	CB	0.2	0.5

LP: Max BA

Subject to:

.45 x1 +.2 x2 - BA > 0

.23 x1 + .5 x2 -BA >0

x1+x2=1

decision variables

x1	0.576923
x2	0.423077
BA	0.344231

OBJ Func 0.344231

Constraints	Used	INEQ	RHS
	0	>	0
	0	>	0
	1	=	1

Our LP solution sates the Mantle should guess FB almost 58% of the time and guess CB 42% of the time and will achieve a BA against Spahn of 0.344. WE find in the dual solution for Warren Spahn that he should throw a FB about 52% of the time and a CB about 48% of the time.

Example 2. We illustrate a 3 x 3 game..

Payoff Matrix

			Colin		
		D	E	F	
	A	9	2	7	x
Rose	B	3	6	4	y
	C	5	3	1	z
		s	t	u	

For Rose, the decision variables are

v = expected value of the game

x = probability for playing strategy A

y = probability for playing strategy B

z = probability for playing strategy C

We formulate the problem as:
Maximize v
Subject to:
$9x+3y+5z-v \geq 0$
$2x+6y+3z-v \geq 0$
$7x+4y+1z-v \geq 0$
$x + y + z=1$
Non-negativity $x, y, z, v \geq 0$

Our solution is $v = 4.8$ when $x = 0.03$, $y = 0.70$, $z=0.0$ and the dual variables are *0.40, 0.60, 0.00* respectively.

Non-Zero Sum games

In a non-zero sum game, the payoff matrix consists of values that all do not sum to be zero. These are called partial conflict games. Here is an example of a non-zero sum game. Our formulation is slightly different and presented in equation (2) and (3):

Maximize V (2)
Subject to:

$$N_{1,1}x_1 + N_{2,1}x_2 + \ldots + N_{m,1}x_n - V \geq 0$$
$$N_{2,1}x_1 + N_{2,2}x_2 + \ldots + N_{m,2}x_n - V \geq 0$$
$$\ldots$$
$$N_{m,1}x_1 + N_{m,2}x_2 + \ldots + N_{m,n}x_n - V \geq 0$$
$$x_1 + x_2 + \ldots + x_n = 1$$

Nonnegativity

where the weights, x_i, yield Rose strategy and the value of V is the value of the game to Colin.

Maximize v (3)
Subject to:

$$M_{1,1}y_1 + M_{2,1}y_2 + \ldots + M_{m,1}y_n - v \geq 0$$
$$M_{2,1}y_1 + M_{2,2}y_2 + \ldots + M_{m,2}y_n - v \geq 0$$
$$\ldots$$
$$M_{m,1}y_1 + M_{m,2}y_2 + \ldots + M_{m,n}y_n - v \geq 0$$
$$y_1 + y_2 + \ldots + y_n = 1$$

Nonnegativity

where the weights, y_i, yield Colin's strategy and the value of v is the value of the game to Rose.

We point out that if the game as a pure strategy solution via movement diagrams or dominance that this must be done prior to using a LP approach. The game must be reduced if it can be reduced.

Example 3: Consider the following partial conflict mixed strategy game.

		Colin	
		C	D
Rose	A	(2,4)	(1,1)
	B	(5,1)	(0,3)

The movement diagram reveals no pure strategy solution. The linear programming formulations for each of our player's in order to find the Nash equilibrium values for Rose and Colin are found as follows formulating the linear programs using equations (2) and (3):

(a) Maximize V_c
Subject to
$4x_1 + x_2 - V_c \geq 0$
$1\, x_1 + 3\, x_2 - V_c \geq 0$
$x_1 + x_2 = 1$
$x_i,\ V_c \geq 0$

(b) Maximize V_r
Subject to
$2\, y_1 + \ y_2 - V_r \geq 0$
$5\ y_1 - V_r \geq 0$
$y_1 + y_2 = 1$
$y_i,\ V_r \geq 0$

The solutions to (a) and (b) are (a) $V_c = 2.2$ when $x_1 = 0.60$ and $x_2 = 0.40$ and $V_r = 1.20$ when $y_1 = 0.25$ and $y_2 = 0.75$. This game results in the Colin playing 1/4 C, 3/4 D and insuring a value of the game of 1.2 for Rose while Rose plays .4 A, .6 B and yielding a value of the game of 2.2 for Colin. The solution is (1.2, 2.2).
The convex polygon and Nash equilibrium is shown in figure 1.

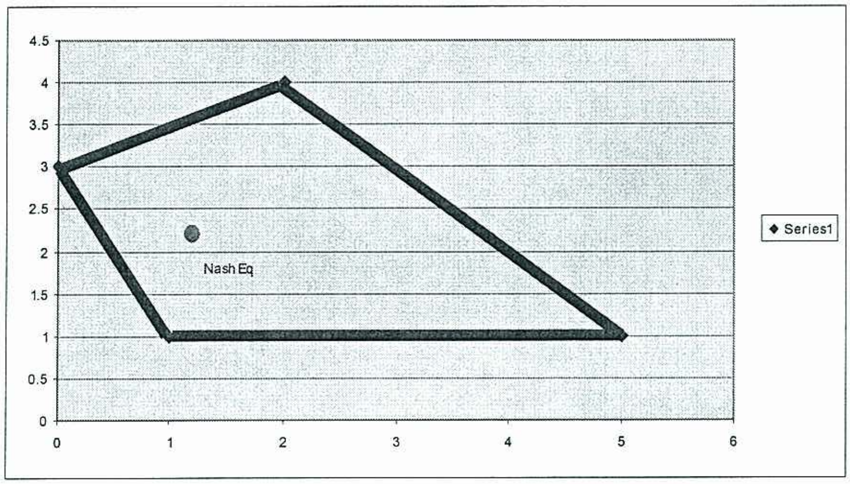

Figure 1. Convex Polygon and Non-Pareto Optimal Nash equilibrium

Nash Arbitration

We quickly present the Nash Arbitration scheme as nonlinear optimization problem. **Nash's Theorem (1950):** There is one and only one arbitration scheme which satisfies Nash's Axioms 1 through 4. It is this: if the *status quo* $SQ = (x_0, y_0)$, then the arbitrated solution point *N* is the point (x, y) in the polygon with $x \geq x_0$ and $y \geq y_0$ which *maximizes the product:* $(x - x_0)(y - y_0)$.

Our approach involves find the constraints that form the convex polygon in figure 1, the security level to act as the status quo point for which our solution has to be at this point or greater, and the objective function: $(x - x_0)(y - y_0)$. With a security level of (1,2.2) we solve the following NLP:

Maximize $Z = (x-1)(y-2.2)$

Subject to: $2x+y \geq 3$

$-2x + y \leq 3$

$y \geq 1$

$x + y \leq 8$

$x \geq 1$

$y \geq 2.2$

The solution by NLP for the Nash arbitration point is *(2.4,3.6)* and the value of $Z = 1.96$. We have found this approach to be excellent as the problems gets more complex.

References

Dorfman, Robert. (1951). Application of the simplex method to a game theory problem. Chapter XXII from *Activity Analysis of Production and Allocation Conference Proceeding*. T. Koopman (ed) John Wiley Publishers. 348-358.

Nash, John (2009). Lecture at NPS. Feb 19, 2009.