PRINCIPLES FOR DESIGNING ACTION-CONSEQUENCE TOOLS IN CALCULUS

Thomas P. Dick
Mathematics Department
Oregon State University
Corvallis, OR 97331
tpdick@math.oregonstate.edu

Key principle for designers: The Action-Consequence Principle

Technology-based learning scenarios should

- allow students to take deliberate, purposeful and mathematically meaningful actions
- provide immediate, visual and mathematically meaningful consequences.

Two key related issues that emerge for designers/authors are

- *Mathematical fidelity* (stay true to the math)

Example: an activity that intends to help students understand the mathematical notion of slope, but uses undirected line segment lengths in a slope triangle is mathematically unfaithful.

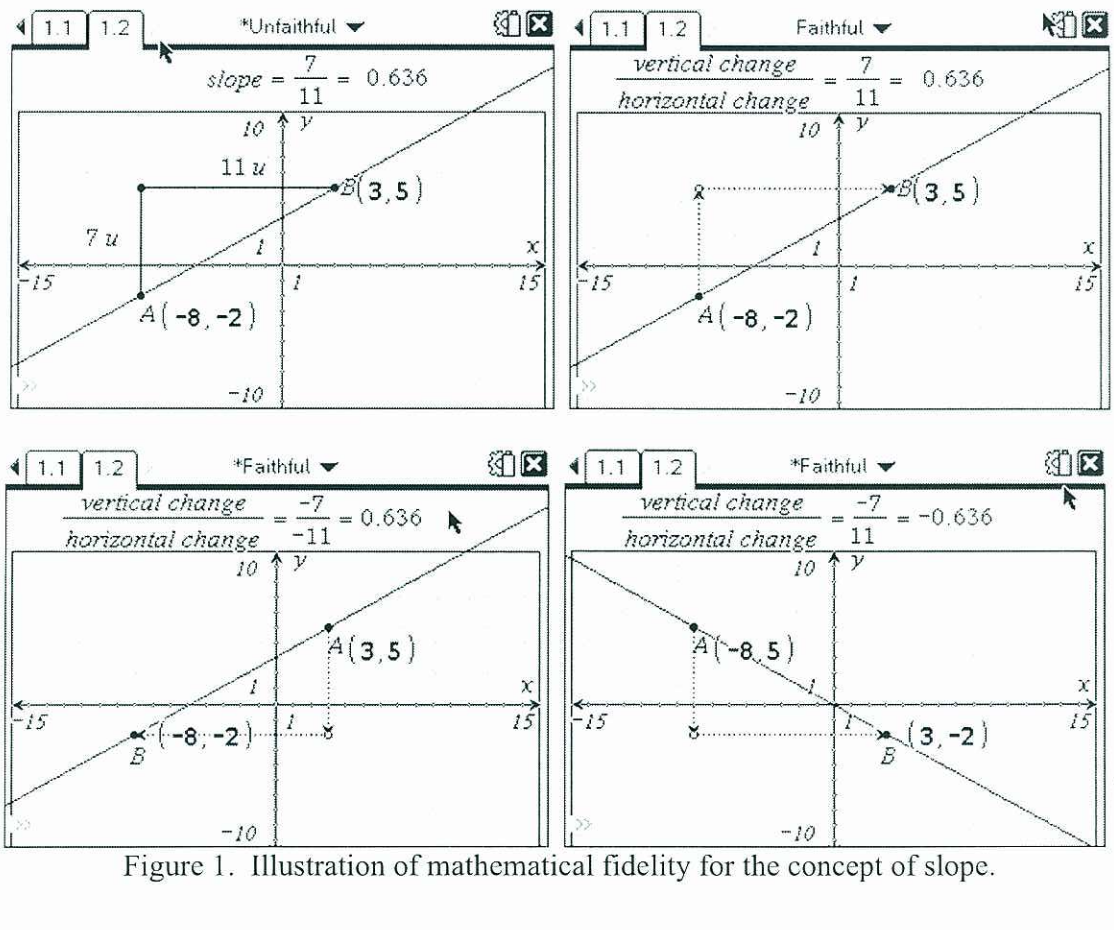

Figure 1. Illustration of mathematical fidelity for the concept of slope.

The screen on the upper left illustrates this example of infidelity. A student using this kind of diagram is "set up" to reason that slope is always non-negative, for the lengths of the legs on a triangle formed on a line falling from left to right would also be positive lengths. The other three screens show a more mathematically faithful rendition of slope, with arrows indicating that the change quantities have direction. This helps make clear (in the second and third screens) that the ratio is invariant for either choice of order of two distinct points on the line, and that the sign of the slope is derived from the signs of the changes in y and x.

• *Cognitive fidelity* (stay true to match cognitive perception and intended math meaning)

Example: Suppose an activity involves a right angle in geometry, but the underlying coordinate system uses unequal scale factors for the axes. This runs the risk of presenting the right angle visually as acute or obtuse, depending on its location, even if the angle measurement is technically "correct."

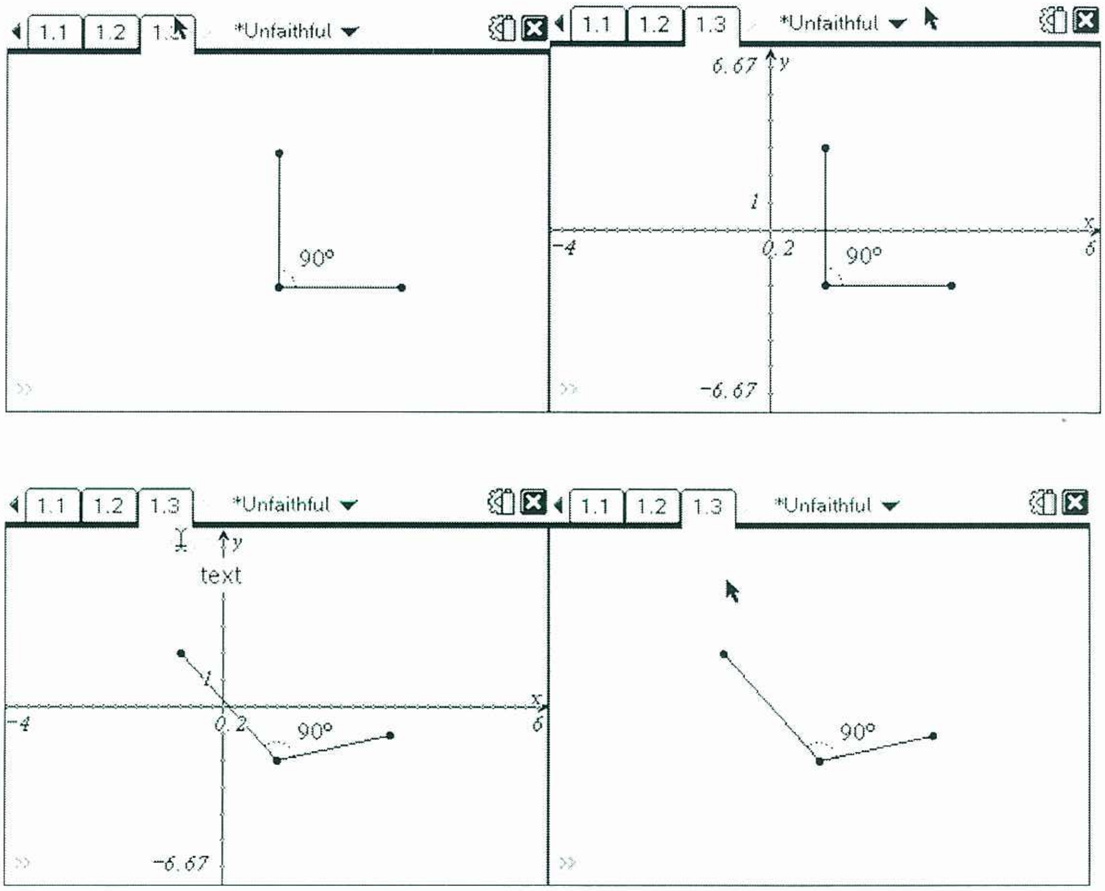

Figure 2. Illustration of cognitive fidelity using angle measurement.

Strict adherence to equal scale factors on the axes rescues the cognitive faithfulness of the representation. Figure 3 illustrates the effect of "squaring" the scaling.

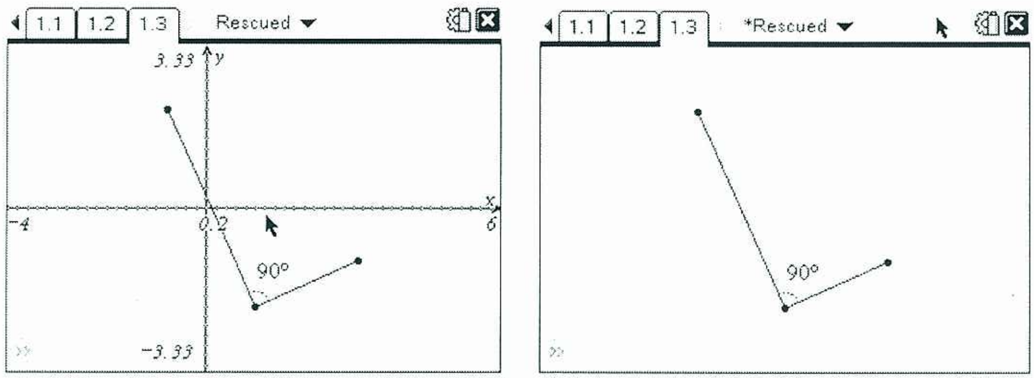

Figure 3. Rescuing cognitive fidelity of angle measure by use of square scaling.

Exploiting Technology: action-consequence scenarios across representations

Dynamic links connect two or more representations so that changes in one are immediately reflected in the others. *Dynamic links* can provide:

1) scenarios for mathematical exploration, 2) immediate visual consequences (feedback), and 3) golden opportunities for inquiry.

Geometry packages have exploited dynamic links within the virtual representation of the Euclidean plane (example: live measurement of the area of a manipulable constructed circle). Relatively recently, this dynamic linking idea has been opened up across other representations, most notably in TI-Nspire. Some have proposed a "Rule of Five" for thinking about representations in mathematics. Here's a picture of such a diagram with every possible arrow shown between representations. If we include virtual enactments of the physical representation with robust and unfettered potential for forging dynamic links, then I can't think of any arrow that is not realizable!

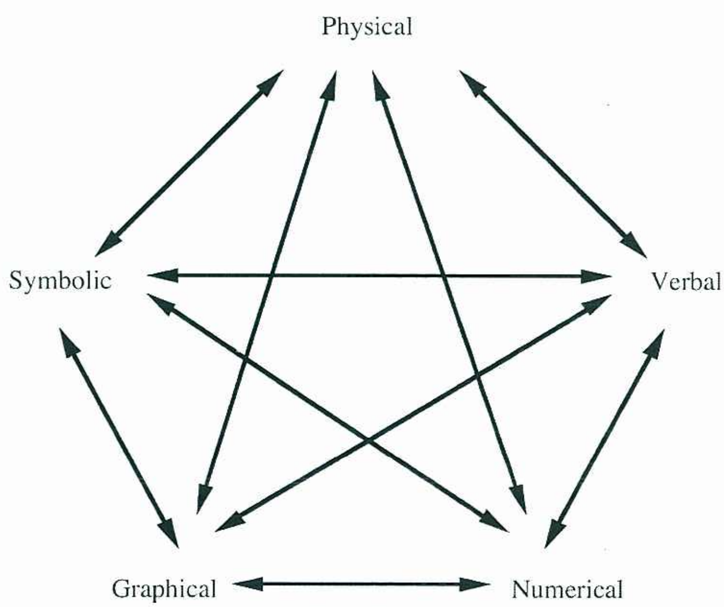

Figure 4. The "Rule of Five".

This diagram can also help us consider dynamic linkages within and between representations. The key idea is to have an action in one representation result in an immediately visual consequence in the same or another representation. A two-way action-consequence scenario is already provided by the graphing environment of TI-Nspire. For example, if one graphs a simple quadratic $y = x^2$ then one can manipulate the graph directly, with changes in the symbolic formula instantly updated. Conversely, one can edit the equation for the quadratic and see an immediate change in the graph. Now consider a dynamically linked derivative graph and you see the potential in calculus.

Examples of Action-Consequence Scenarios for Calculus

1. A Tale of Two Lines

This is a classic use of graphing to illustrate a famous theorem in calculus. Consider the screen on the left, showing two lines with a common x-intercept, say at $x=a$. What is the ratio of the y-coordinates for any other value $x \neq a$? What is the ratio of the slopes of the two lines? It is not hard to see that they are the same.

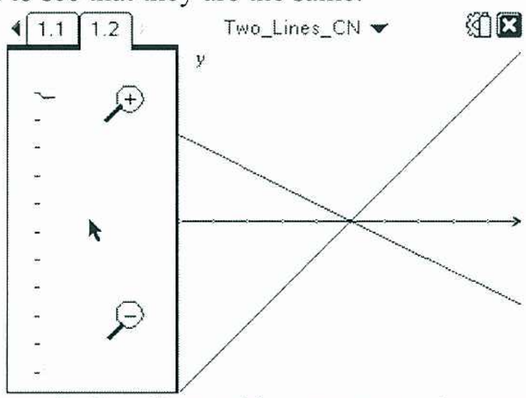

Figure 5. Two lines with a common x-intercept?

But the map zooming icons to the left of the graph suggest this is a "zoomed in" view. Let's zoom out and see the next screen.

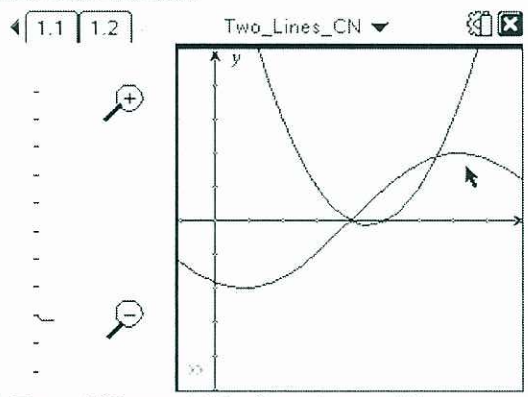

Figure 6. Two differentiable functions with a common zero!

But the map zooming icons to the left of the graph suggest this is a "zoomed in" view. These were not two lines after all, but rather, two differentiable functions with a common zero. As x approaches a we can see that the ratio of the function values and the ratio of the derivative values approach the same limit: L'Hopital's Rule!

2. The filling urn

This is an adaptation of a graphical interpretation task found in many international curricula. Indeed, it can be used at grade levels well below calculus, but it also lends itself well to talking about related rates, and first and second derivatives in context.

The action-consequence screen shows an urn of a given shape with a corresponding graph of the fluid level in the urn as a function of the volume of fluid in the urn. The clicker at the top of the screen provides the action – each click adds one unit (of volume) of liquid to the urn, with both a physical and graphical depiction of the corresponding current fluid level.

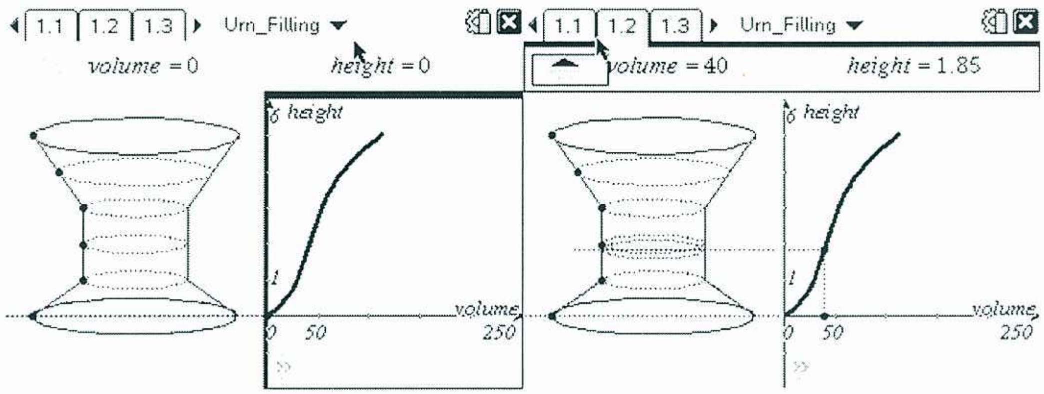

Figure 7. The filling urn – representation of fluid level physically and graphically.

But that is not all! The shape of the urn is dynamic also. By grabbing any of the vertex handles on the left of the urn, one can change its outline, with the corresponding graph changing immediately. This allows additional kinds of questions, such as, "what shape of urn would result in a perfectly linear graph?" Some students might guess a conical shape, but this prediction is easily tested and provides a great opportunity for a discussion of why that graph is actually concave down.

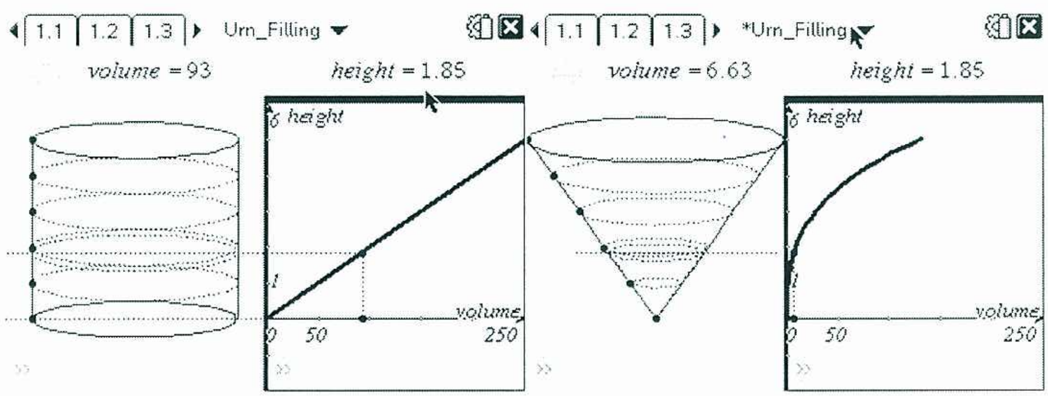

Figure 8. The filling urn – shape can be dynamically changed.

3. The elevator

These documents allow for an exploration of velocity and position in the context of the vertical motion of an elevator. Starting out with vertical motion has its advantages over the usual illustration of rectilinear motion in terms of the horizontal position of a car. Namely, the association of position on the vertical axis of a graph associates directly with the physical vertical position of the elevator. The two screens below illustrate the scenario. The physical position of the elevator is shown along with a graph of its position as a function of time. The clicker in the upper left advances time. The position of the elevator is governed by the position function (which can be specified by the user), and both the physical location of the elevator and the corresponding point on the graph are driven by the clicker. The second screen shows the corresponding velocity function for the same motion, and the third screen shows both graphs at the same time, with the time axes aligned.

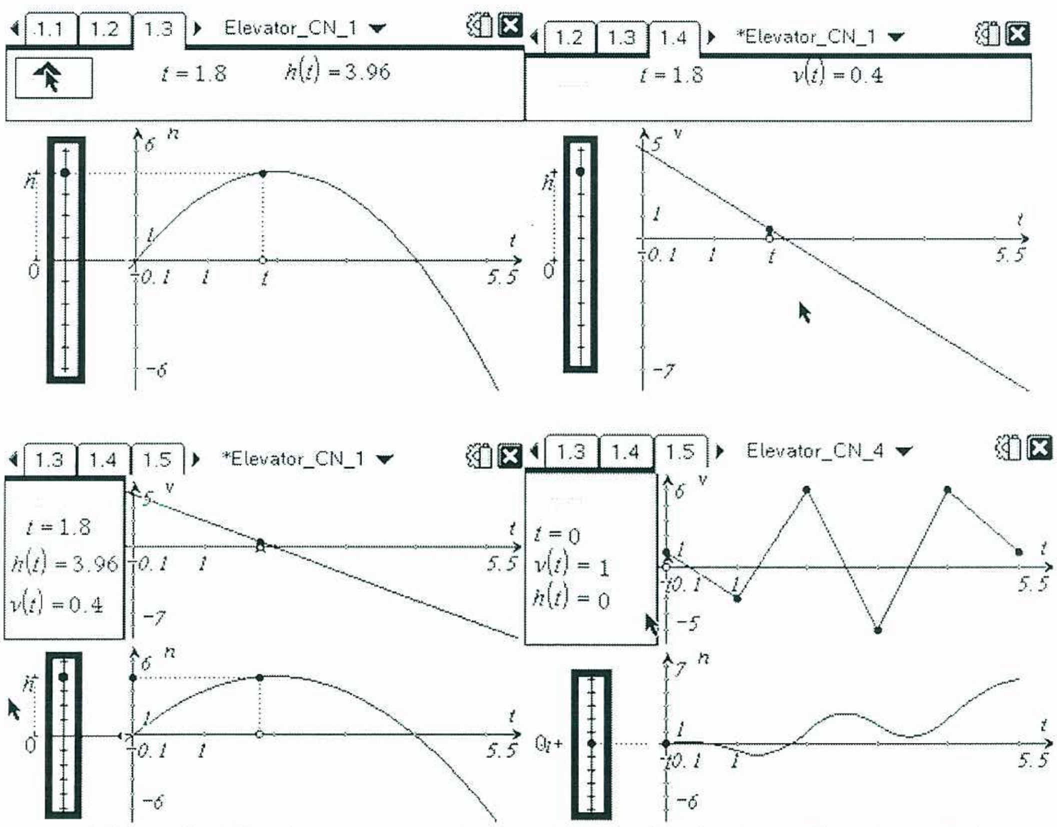

Figure 9. The elevator – vertical position and velocity as functions of time.

The fourth screen shown above comes from another document in this family. In this case, it is the *velocity* function that is specified by the user, but by a directly manipulable piecewise linear function. Under the assumption that the initial position of the elevator is ground level ($h = 0$), the position function is dynamically generated!

4. The Taylor polynomial grapher

This document provides new opportunities beyond those provided by a graphing calculator. The user can specify a function and both the degree and the center $x = a$ of expansion of the Taylor polynomial are dynamic. The integer degree is controlled by a clicker in the upper right of the screen. The center of expansion is a draggable point on the x-axis. Screens 1, 2, and 3 below illustrate the first (tangent line), second, and third degree Taylor polynomials for the same point $x = a$.

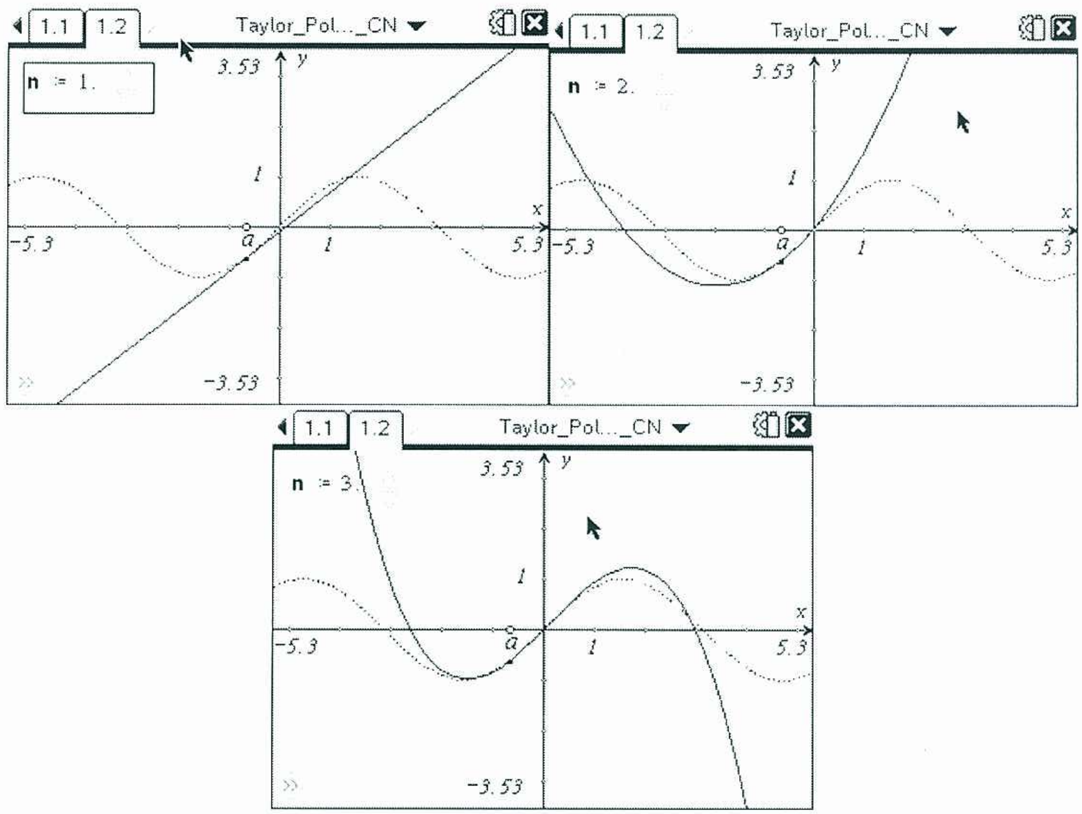

Figure 10. Taylor polynomials of degree one, two, and three

If one sets the degree to $n = 2$ and now drags the point $x = a$, one obtains a dynamic "quadratic of best (local) fit" – in essence, an iconic concavity detector! The next 2 screens illustrate the use of this detector near a local minimum a local maximum, while the third screen shows a situation where the quadratic of best fit becomes linear. Indeed, by dragging across this point, one sees the parabola transition from opening up to opening down – an inflection point. The leading coefficient of the quadratic of best fit is determined by the second derivative. The transition to a tangent line at this special point indicates that the second derivative vanishes there.

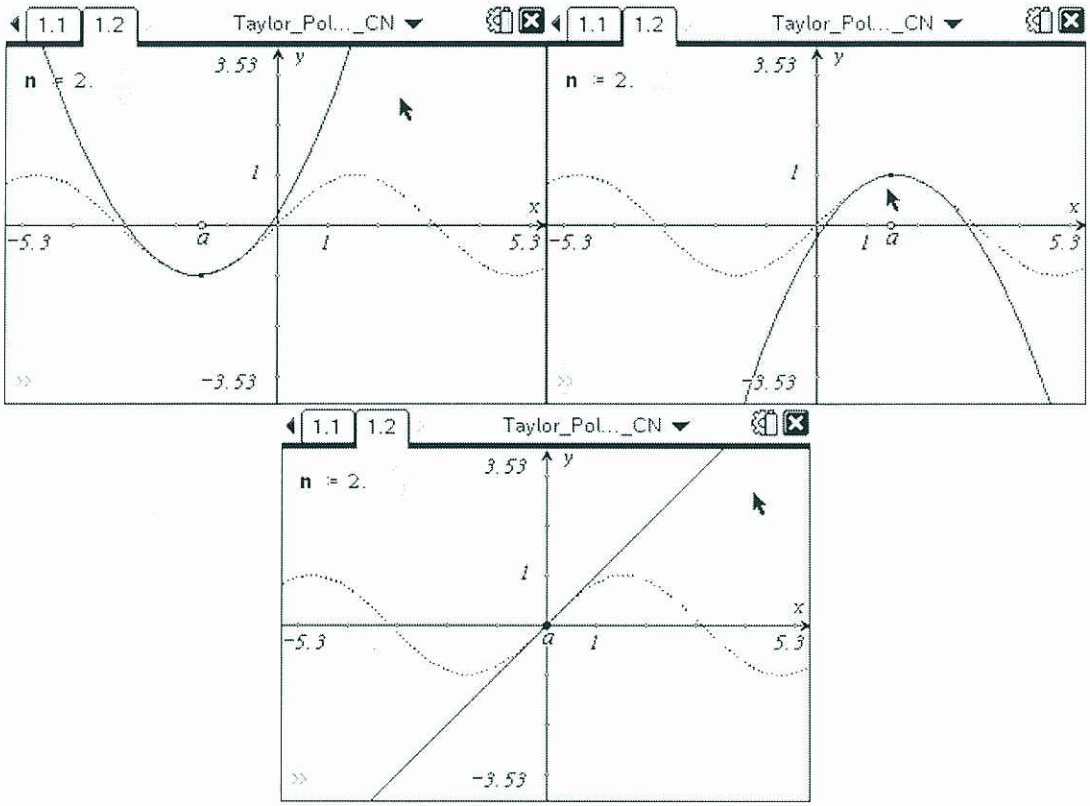

Figure 11. Quadratics of best fit

Key principle for teachers: Ask good questions!

A good action-consequence scenario lends itself to good sense-making questions:

- Predict consequence in advance of action (what would happen if…?)
- Consider action to produce a given consequence (what would make … happen?)
- Conjecturing/Testing/Generalization (When…?)
- Justification (Why…?)

Therein is the key issue for teachers: *ask good questions!* Even the most compelling action-consequence environment can be rendered ineffectual if teachers use it to simply prescribe actions for students and direct them to just record the observed consequences.

The bottom line: it is critically important that we pose questions that demand reflection, sense-making, and reasoning.