CAS AS A RESTRUCTING TOOL IN MATHEMATICS EDUCATION

Keith A. Nabb
Moraine Valley Community College
9000 West College Parkway
Crawley Science, Room 242
Palos Hills, IL 60465
nabb@morainevalley.edu

INTRODUCTION

The term *computer algebra system* (CAS) is generally used to describe the diverse class of technological tools equipped with numerical, graphical, and symbolic capabilities. These tools can appear as computer software programs such as *Mathematica* (Wolfram Research, 2009) and *Maple* (Waterloo Maple, 2009) or as graphing calculator interfaces such as the TI-92, Voyage 200, or TI-Nspire (Texas Instruments, 2009). By and large, they are assumed to hold great promise in enhancing mathematical teaching and learning (Blume & Heid, 2008; Fey, Cuoco, Kieran, McMullin, & Zbiek, 2003; Heid & Blume, 2008; NCTM, 2000; Zbiek & Heid, 2009). Apart from standard graphing calculators which possess only numerical and graphical functionality, it is the *symbolic* capacity of CAS and its connectivity with numerical and graphical functionalities that has captured the attention of researchers and teachers internationally.[1]

Given the potentials of such sophisticated instruments, a fundamental question has emerged concerning its presence in mathematics classrooms. Specifically, how should such devices be used in the teaching and learning of mathematics? Some espouse that the potential of CAS rests in "freeing" students from mundane drills so that increased energy may be channeled into thinking and reflecting on the mathematics learned. Those opposed caution that such use threatens the acquisition of basic skills necessary in the learning of future mathematics. Still others take a neutral position in remaining optimistic about CAS but are troubled by the technical challenges that teachers and students are likely to encounter. Regardless, the multitude of positions taken on these issues provides fertile ground for debate with respect to CAS in mathematics education.

LITERATURE SELECTION AND ORGANIZATION

The body of research concerning computer algebra use in math education is voluminous. It contains a variety of information sources including literature reviews, theoretical pieces, systematic research studies, and opinion papers. Although it could be argued that most research domains are compartmentalized as such, the CAS research is especially splintered in this respect. Specifically, efforts to connect research and practice seem to lag behind in comparison to other research domains (Zbiek, 2003). Moreover, because the field is generally considered to lack cohesion with respect to unifying theories (Zbiek,

[1] By *symbolic capacity*, researchers generally speak of the ability of CAS to manipulate algebraic expressions, test for equivalence, generate answers in exact form, and the like.

Heid, Blume, & Dick, 2007), I have made no attempt to intentionally exclude research based on its classification or origin. The reader will find the literature to be of an international flavor, with heavy influences from Austrian, Australian, French, and North American authorities on CAS.

The organization of this paper is as follows. To begin, a discussion of important theoretical contributions will be provided. Since many of the studies discussed in this paper embrace one or several of these perspectives, it is important to provide the reader some background information on such theories. Next, the paper turns to the central question of CAS use. Amid this discussion, important findings and drawbacks are gathered from this work through the eyes of the theories previously mentioned. Finally, a model of CAS use concludes the paper.

THEORETICAL DEVELOPMENTS

Instrumental Genesis

More than any single theoretical construct in the literature, the idea of *instrumental genesis* has served as the underpinning of research on mathematical learning in CAS environments. Emanating from the work of Vérillon and Rabardel (1995), this idea asserts that using any tool—albeit a hammer, a drill press, or a computer algebra system—is rarely spontaneous and automatic. A key factor here is the distinction between the *artifact* (the tangible manmade object) and the *instrument* (the psychological tool used in acts of learning). It is only once the user has been able to adopt the physical artifact for a meaningful purpose that this genesis begins to unfold: "…a machine or a technical system does not immediately constitute a tool for the subject. Even explicitly constructed as a tool, it is not, as such, an instrument for the subject. It becomes so when the subject has been able to appropriate it for himself." (Vérillon & Rabardel, 1995, p. 84-85). Although Vérillon and Rabardel make no mention of computer algebra and only occasionally reference math/science education, the widespread application of their theory has cast much light on the field. It has served in explaining many of the potentials and pitfalls of adopting, implementing, and understanding the impact of technology on students' mathematical thinking.

Cognitive Technologies: Epistemic or Pragmatic?

The view of CAS as a computation tool whose sole purpose is to solve mathematics problems is fortunately not the widely espoused view. Researchers are generally concerned with CAS and its intrinsic value on the educational experience—specifically, its ability to play a role in students' learning and understanding of mathematics. Pea (1987) used the term "cognitive technology" to convey the idea that such technologies can assist the user in "learning how to learn" (Pea, 1987, p. 111). These cognitive tools can leave traces of student work, foster reflection on such work, elicit what-if scenarios, and orchestrate other formative means of thinking-in-action. Pea interprets the term *technology* broadly as any invention that has provided the means for future advancement in a civilized society (e.g., symbols, written language, theories, artifacts, and the like).

His central thesis is that computer technologies can explicate internal thinking processes, and, in turn, provide the learner a tangible means of reflection. He makes quite clear that the *cognitive* potential of computers as technologies sets them apart from other technologies. For example, although a pencil may come to one's aid in reproducing a memorized list, it does so exclusively in an organizational way. In no way does the pencil stretch mental capacity.

Having benefited from the above perspective, researchers have examined the dual affordances of CAS—specifically, (a) the efficient machine output of mathematical solutions, and (2) the genuine reflections from CAS that augment learning experiences. The constructs used to frame these affordances are the *pragmatic value* and *epistemic value* of mathematical activities with technology (Artigue, 2002; Lagrange, 1999; 2003; Ruthven, 2002). A technique's *pragmatic value* centers on its ease of use and efficiency in accomplishing a task while its *epistemic value* concerns its potential to enrich the user's understanding of the mathematics at hand. For example (even in the absence of technology), using a highly routine mathematical procedure may allow the learner to bypass thinking; this is indeed pragmatic but has low epistemic value. Artigue (2002) argues that the use of CAS in mathematics classrooms results in an imbalance to this didactic model: "Techniques that are instrumented by computer technology are changed, and this changes both their pragmatic and epistemic values." (p. 249). Debates in this regard are often filtered through the conceptual and technical aspects of the activity. Because these ideas are a mainstay in the literature, it is to these aspects of the theory that we now turn.

The Technical/Conceptual Divide

In early North American studies, there appeared a predominant theme that CAS could be used to outsource tasks of mathematical drill so that students could focus on crafting solution methods and interpreting results (Heid, 2003). Specifically, some studies called into question the widespread view that procedural mastery need precede conceptual understanding (Heid, 1988; Palmiter, 1991). In light of these findings, an upheaval to traditional mathematics curricula seemed imminent.

Although skepticism was widespread, two important findings were vital to easing the concerns of CAS's "intrusion" in mathematics. First, it was found that CAS use does not, in general, weaken students' abilities to perform routine algebraic manipulation (Ayers, Davis, Dubinsky, & Lewin, 1988; Heid, 1988; Hillel, Lee, Laborde, & Linchevski, 1992; Palmiter, 1991). Even more important, this general finding transcends multiple grade levels as well as ability levels (cf. Heid, Blume, Hollebrands, & Piez, 2002). Second, students' conceptual growth and understanding are not lessened as a result of CAS use (Heid, 1988; Judson, 1990). In fact, O'Callaghan (1998) found learning in a CAS environment to foster deeper conceptual connections of functions in comparison to similar learning in the absence of CAS. Despite these findings, teachers' marginal use of computer algebra (Artigue, 2000) may be explained in some measure by the fact that having "no effect" or "minimal harm" is hardly a reason to change: "Unless an improvement occurs in some aspect of mathematics learning [with CAS], the argument for change is not compelling." (Heid et al., 2002, p. 587).

Other research findings (Guin & Trouche, 1999; Lagrange, 2003) alongside reflective commentaries (Artigue, 2002; Ruthven, 2002) suggest that the perceived usefulness of CAS in supplanting the technical in favor of the conceptual may be overstated. For example, some studies have found that technology does not invoke reflective thinking on its own (Guin & Trouche, 1999; Hoyles & Noss, 1992) while others remind us that using CAS is challenging in and of itself for students and teachers (Drijvers, 2000; 2002; Lagrange, 2003). Together, this shines the spotlight on new aspects of technical skill that consume the user.

Constraints, Boundaries, and Obstacles

The important work in the areas of instrumentation and instrumentalization (cf. Artigue, 2002; Guin & Trouche, 1999) has fueled researchers to explore the barriers posed by CAS. Anytime a user's actions are constrained by or filtered through a learning tool, the danger of magnifying the specificity of the content learned is very real (cf. Hoyles et al., 2004). The result is an added challenge for users in mapping this situated knowledge to the broader knowledge they are trying to acquire. This is a view generally embraced by Drijvers (2000, 2002), Guin and Trouche (1999), and Hoyles et al. (2004). Given the centrality of such a concern, it further supports the importance of the teachers' role in instrumental orchestration (Guin & Trouche, 2002); that is, students may encounter additional technical difficulties with CAS and need guided assistance in moving past such barriers.

Drijvers (2000, 2002) has been a leader in identifying obstacles that students are likely to encounter in CAS environments. He defines an obstacle as "a barrier provided by the CAS that prevents the student from carrying out the utilization scheme that s/he has in mind. As a result, the obstacle stops the process of shifting between the 'pure' mathematics and the problem situation." (Drijvers, 2000, p. 195). Common obstacles revealed in his work include how one copes with (1) unexpected/ill-conceived output, (2) the seemingly arbitrary discretions of CAS, and (3) CAS's refusal to execute commands. The above obstacles are centerpieces in discussions of the 'black-box' nature of CAS (Bossé and Nandakumar, 2004; Buchberger, 1989, 2002; Child, 2002; McCallum, 2003) while other obstacles might be considered more global in nature (Drijvers, 2000, 2002). In later work, Drijvers (2002) documents obstacles that arise when the technical and conceptual components of an activity clash "...either because the technique is not accompanied by appropriate meaning and conception, or because the technical skills for the performance of a conceptual idea are lacking." (p. 224).

CAS USE IN MATHEMATICS CLASSROOMS

The words of Heid, Hollebrands, and Iseri (2002) convey a central dilemma concerning the use of CAS: "What place does this powerful technology have in our classrooms?" (p. 210). Although the question appears simple on the surface, many research studies have cast light on the issue only to reveal little common ground. In this section, five distinct uses will be discussed: black box, white box, amplifier, discussion tool, and catalyst for reform. The aim is to bring to light these different perspectives as well as the reasons why specific uses may be favored in the long run. The section closes with the observation that these uses may be viewed in the form of a nested model ranging from rudimentary use with straightforward implementation to a sophisticated use whose potential is only beginning to be understood.

Black Box

The use of CAS to produce answers to mathematical questions with no attention to reasoning has received widespread criticism in mathematics education. The term "black box" (relative to computer algebra) was introduced by Buchberger (1989) to convey precisely this use—CAS as another authority figure in the classroom generating results sans the *how* or *why*. Without knowledge of the underlying mathematics, many agree that the consequences of such use are disastrous for education and beyond (Bossé & Nandakumar, 2004; Buchberger, 2002; McCallum, 2003). With only a few keystrokes, CAS may display inconsistent, unpredictable and even erroneous results as interpreted by the user. In a real sense, CAS hijacks the user's input and performs mysterious and sometimes unintended operations. Of course, the bright side of "black box" usage is the spark of curiosity it instills in some students (Boyce & Ecker, 1995; Heid, Hollebrands, & Iseri, 2002) but otherwise, one must be careful not to cultivate an anxiety-inducing view of a subject already conceived as mysterious (cf. Paulos, 1988). Finally, the black box approach can serve the purpose of solving problems that literally stretch human capacities to their limits—the message being that CAS can handle exceptionally intricate problems. Regardless, "black box" has relatively low epistemic value and appears to add little to the educational experience as a whole (Artigue, 2000).

White Box

Critical of the above use of CAS, many researchers and practitioners advocate the *informed* pedagogical use of computer algebra—what has become known in the literature as "white box" (Buchberger, 1989; Child, 2002; McCallum, 2003). For example, Heid and Edwards (2001) discuss white box usage in the context of solving linear equations. Given an equation such as $3x-4=7$, a student may add four to both sides ($3x=11$) but then decide, somewhat prematurely, to subtract 3 from both sides. Were this suggestion offered in an ordinary classroom setting, it would likely be squashed in favor of automated suggestions such as "divide by three" or "multiply by one-third." In short, a potential learning experience would be lost. CAS, on the other hand, will execute precisely the student's request (Figure 1, left) so a learning experience awaits the user even amidst this suboptimal move. Figure 1 (right) illustrates what the learner might do after this realization.

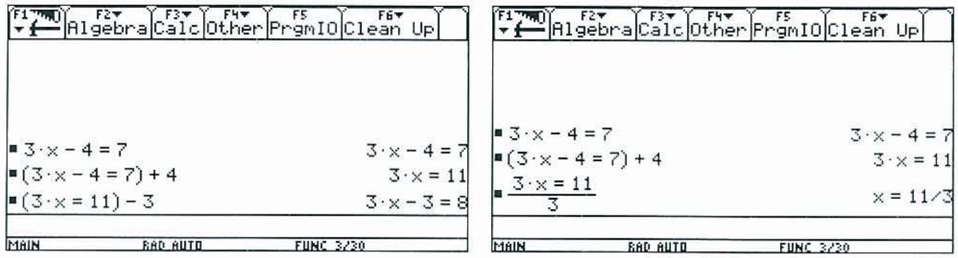

Figure 1. White Box

In light of the above example, Heid and Edwards (2001) tout that it is CAS's ability to give instantaneous and nonjudgmental feedback that opens doors for novices who are struggling with concepts. Dick (2008) offers similar remarks with respect to the learning of calculus. While black box usage in evaluating $\int x^2 \ln x\,dx$ may benefit a student to some degree, a pop-up window in which the user selects u and dv (in an application of integration by parts) is far more likely to stimulate higher-level thinking.

Even in non-CAS settings (e.g., students working with the programming language *Logo*), the pedagogical use of computers has been shown to offer genuine learning opportunities. Hoyles and Noss (1992) discuss a particularly illuminating episode where a thirteen year old student comes to understand that multiplication by a small number (between zero and one) decreases the original number. Although the authors mention several aspects that contribute to this learning, it is the immediate feedback from *Logo* that fosters refinements in the student's thinking and, consequently, adjustments to previous attempts to reach a goal. Conventional pencil-and-paper techniques seriously hinder this process. Consonant with the view of Heid and Edwards (2001), it is the immediacy and neutrality of the computer's responses that offer fertile ground for learning.

Amplifier

CAS can serve the role of amplifier to intellectual activity. That is, computer algebra tools can produce many varied examples in quick succession to the effect of offering assistance in discerning regularities which might otherwise remain hidden (Heid, 1997; Pea, 1987). It can also serve as a general experimental tool as one delves into the unknown world of mathematics. Generally speaking, such uses relegate "manual labor" (e.g., plotting points, repetitive multiplication, etc.) to a sometimes invisible level so that users may step back and generalize on a broader scale. This intrinsic attribute of "outsourcing procedures" to CAS is often equated with the amplifier role (Arnold, 2004; Heid, 1988, 1997, 2003; Heid & Edwards, 2001; Kutzler, 2003; McCallum, 2003; Palmiter, 1991).

The reader might find it surprising that there is little research to support amplification as putting students in a better position to learn mathematics. The reasons for this are two-fold. First, Pea's (1987) original work highlights cognitive technologies in the broadest sense—programming languages, algebra systems, geometry software, and intelligent tutors. Vast amounts of the research literature cite amplification as an important use of

CAS but this is almost always used to steer the discussion toward changing teacher practice or curriculum (cf., Heid, 1988; Palmiter, 1991). Second, the amplifier metaphor tends to be particularly well-suited to students' generalizations in graphing environments. For example, a student may graph three or four members of a family of functions and formulate conjectures with respect to the changes on screen. Thus, the halfhearted attention to amplification can be attributed to standard graphing calculators and their ability to perform these functions just as well. Since graphical excursions minimize CAS's most prized possession—algebraic manipulation—this finding is not surprising.

Discussion Tool

The externalization of mathematical ideas to foster dialogue in classroom settings is a mainstay in CAS research (Guin & Trouche, 1999, 2002; Heid, 1997; Pea, 1987). Pea (1987) asserts that cognitive technologies "make *external* the intermediate products of thinking . . . which can then be analyzed, reflected upon, and discussed." (p. 91). A nice example of this functionality can be found in the work of Pierce and Stacey (2001). This study examined 30 students in Australia as they took an undergraduate course in calculus in which the CAS *Derive* was integrated. Although the researchers' aims included examining students' flexibility in representations through amplification, the authors were particularly interested in whether CAS prompted meaningful mathematical discussions. When students were asked about whether conversations took place while sharing a computer, 74% of the responses were either "very often" or "always." One student's perspective on this issue is especially illuminating: "In the computer labs we get together as a group. Something will happen on one machine and everyone will go and look and talk about it." (Pierce & Stacey, 2001, p. 37). Although the authors cite the novelty of computers and computer algebra systems as likely contributors for such enthusiasm, the growth in CAS-related research suggests that its presence is a catalyst in initiating discussions of what appears on screen (Boyce & Ecker, 1995; Drijvers, 2003; Kieran & Drijvers, 2006).

Using a more direct approach to spark discourse, Guin and Trouche (1999) examined students' development/evolution of strategies with a novel physical classroom arrangement. Each day, a different student's CAS calculator (TI-92) was connected to a large projector for the whole class to view (even though every student had his/her own calculator). This special student, called the "sherpa student," played a central role in the lesson by assisting the teacher with lesson content and syntactical issues for his/her classmates to view. It is mentioned that this format fostered an environment of open discussion and debate in two ways. First, the small calculator screen—often personal and private to the user—was on public display for discussion. The dialogue was multifunctional in addressing mathematical, syntactical, or otherwise peculiar aspects of CAS. Second, the physical format of the environment enforced social norms of mathematical activity that were predicated on free and open dialogue. Additional research (e.g., Boyce & Ecker, 1995) provides evidence that CAS can be especially fertile in promoting and orchestrating meaningful discourse, even in cases where the teacher is the sole user of CAS.

Catalyst for Reform

Broadly speaking, reform in education might be defined as any movement that results in a nontraditional approach to learning a subject, irrespective of whether change is channeled through teaching or student activity. Research that highlights this transformation explicitly includes the works of Heid (1988) and Palmiter (1991). For example, Heid (1988) utilized a "concepts first" curriculum in which a group of students used CAS in the learning of calculus concepts, postponing skill-oriented mastery until the final three weeks of the semester. Meanwhile, a control group learned calculus in the traditional sense of blended skills and concepts. The results of the study showed no significant difference between the groups with respect to procedural mastery. This outcome *directly* challenges the assumption that procedural fluency need precede conceptual fluency in the calculus curriculum. Even if this assumption is not explicitly embraced by mathematicians or teachers, it is deeply woven in the fabric of the K-16 curriculum.

Finally, in a study of the use of *Maple* in remediation, Hillel et al. (1992) remark on the necessity of making sequential changes (and omissions) to a course in order to accommodate for the presence of CAS. Specifically, due to the wide array of situations that *Maple* treats uniformly, the authors found congruence in using a general approach to teaching functions. This clashes with the traditional hierarchy of first introducing lines, followed by quadratics, then polynomials, etc., as would be considered standard in mathematics curricula: "…a student using *Maple* can analyze $x^2 \sin x$ just as easily as x^2 if taught what aspects of the behavior of functions are useful to look for." (Hillel et al., 1992, p. 136). This research coheres with other studies (Heid, 1988; Judson, 1990; Palmiter, 1991) that emphasize (a) conceptual growth through interpretation and (b) atmospheres conducive to experimentation and conjecture (cf. Kutzler, 2003).

A Model of CAS Use

Given the variety of uses discussed in this section, it is helpful, if for organizational purposes alone, to rank the spectrum of CAS utility in a way that synthesizes this multiplicity. For example, the literature reveals that black box usage offers little to learners' conceptual growth but it is precisely for this reason that this use is both uncomplicated and commonplace. In general, it appears that the black box poses a minimal threat to the "traditions" of mathematics teaching since it serves chiefly as a secondary authority figure. Quite the contrary, using CAS to reshape mathematical activity/pedagogy requires greater teacher innovation in concert with students' emerging cognitive needs. This latter role redefines the status quo, sometimes resulting in an allegiance to specific CAS use(s) or a profound skepticism toward the loss of "classical content." Embracing the view that something may be gained from each of the uses discussed here, a rudimentary model is proposed below.

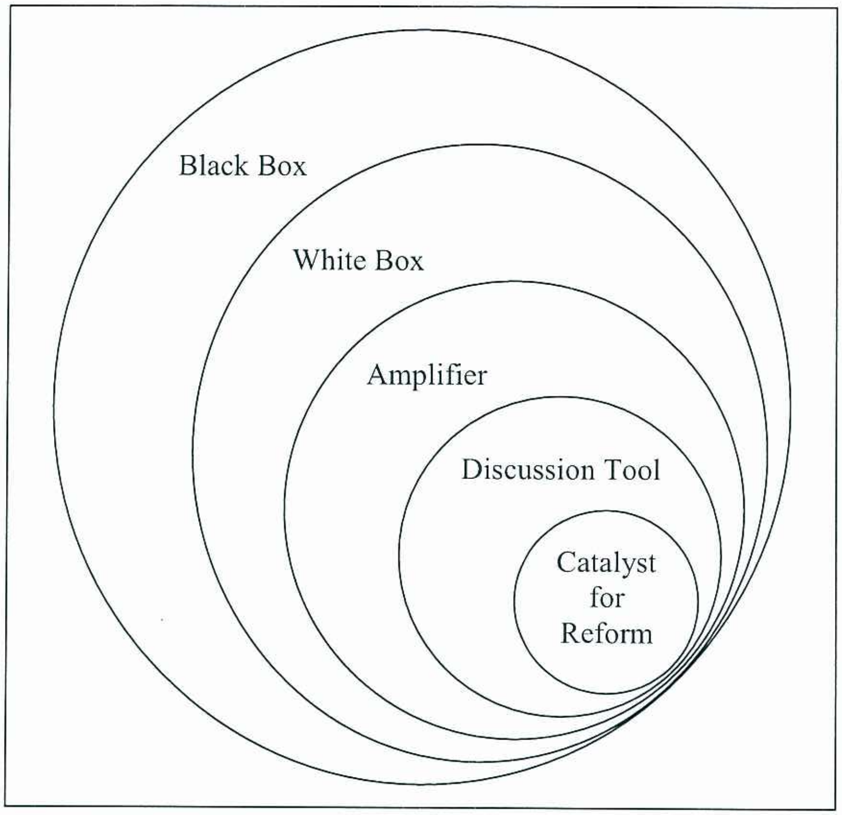

Figure 2. A Model of CAS Use

The nested model illustrates that CAS use at any level likely subsumes its less sophisticated uses. For example, using CAS as an amplifier takes advantage of both pedagogical tool (white box) and black box (de Alwis, 2002). On the other hand, using CAS *solely* as a black box may not—in any conceivable way—incorporate any of the other uses discussed in this paper (Buchberger, 1989). Additionally, the sizes of the circles in Figure 2 are meant to convey (albeit crudely) both the category's ease of implementation and degree of presence in mathematics classrooms. Generally speaking, successively smaller circles signify decreased popularity of such use—most likely a function of the thoughtful purpose and investment needed to make this a classroom reality. At some point in the future, it would be interesting to investigate how learners interpret the influence of these uses on their knowledge of mathematics, as well as the specifics of instrumental genesis in the adaptation of such uses.

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