Learning Labs for Linear Algebra

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Abstract

We introduce a set of twenty learning labs for a linear algebra course. These labs are designed in Maple and all calculations are done in an algorithmic fashion using existing procedures in the Maple kernel. Each lab has an introductory pre-lab component explaining the concept. The lab portion begins with a warm-up problem followed by further computations in Maple. Experiential learning is at the heart of these labs since students explore linear algebraic concepts through the use of a computer algebra system. They work in groups of two or three, each with their own laptop, so that these learning centered clusters are able exchange ideas in the spirit of cooperative learning. In this article, we will discuss the *raison d'être* of lab design and explore one of them. These interactive learning labs are easily adaptable with minor modifications to fit most introductory linear algebra courses.

1 Introduction

There are few courses in mathematics which highlight the connections mathematics has to other disciplines. Linear Algebra with Applications is the best example of such a course. For mathematics majors, this is the course that first exposes them to abstract proof, and for science and engineering majors this course offers necessary mathematical tools that are useful in various applications. Linear Algebra is also a course that is required for education majors as well as economics and finance majors. The purpose of this article is to introduce a collection of twenty learning labs, purposefully designed, to create a learning environment for a first course in Linear Algebra to become a conduit that channels students into various disciplines which entail a firm mastery understanding of mathematical objects.

Properly implemented pedagogy will facilitate a pervasive and smooth dissemination of mathematical concepts in redesigned mathematics courses. It is then reasonable to raise a question about appropriate pedagogical techniques for such redesign. For the purposes of our redesign, we will make a strong case for experiential learning delivered on a technology platform (learning labs) and demonstrate that these pedagogical techniques are the corner stones upon which new and improved linear algebra curriculum should be implemented.

Most of our students who complete the first course in Linear Algebra follow it with a senior level Theory of Matrices course. It has been a long standing practice to prepare students from the very beginning to succeed in both these courses. With this philosophy in mind, we have adopted the David Lay text [7] for the Linear Algebra course, due mainly to its treatment of linear transformations at the very onset and for the excellent collection of application problems in it. We have not had the same success with a traditional text for the Theory of Matrices course. To remedy this situation, we have written an electronic textbook, a truly interactive PDF document, the latter half of which could be used as a companion text for the second course. The first six chapters mirror the material covered in Linear Algebra and the next six chapters cover the extended topics in matrix theory. The e-book gives students the advantage and flexibility in choosing material for reading and cross referencing at the convenience of a mouse click. The e-book is offered to students in both courses at no cost. In the redesign of our Linear Algebra course, we added two instructional components to the existing curriculum.

1. Twenty mini-videos that guide a student through a short activity that traces a pathway to a concept that will be covered in a subsequent Learning Lab.
2. Twenty Learning Labs in Maple to be completed in class following the day after the students have previewed the associated mini-video.

With the advent of these two pedagogical enhancements, we envision the following consequences. Having continued access to the mini-videos will facilitate self-paced learning, whereby a student can complete the assignments at a time when optimal learning has occurred. The Learning Labs will simulate interactions within a normal classroom environment creating a sense of *esprit de corps* among lab partners [5]. A major implication of these advancements is that, Linear Algebra will become a valuable mathematical tool for students and not a barrier that they need to overcome. At the heart of it all is experiential learning. The experiential learner, according to David Kolb [6], goes through a cyclic learning process which recurs in stages. An ideal learner cycles through these learning modes, and never actually settles for a single orientation. Because we have all types of learners in our classroom [5], who are most likely not ideal learners, we must use pedagogical practices that are suitable for all learners.

2 Experiential Learning through Cooperation

Experiential learning is a theory that advocates the idea that students learn from experiences. Effectual learning occurs within a cooperative learning framework that encourages the exchange of ideas. Experiential learning theory offers a foundation that is soundly based in a cognitive process that balances a students' education, personal development, and work. The experiential learner, according to David Kolb [6], goes through a cyclic learning process that develops in four stages. First the learner encounters *concrete experiences* (CE), followed by *reflections and observations* (RO), which directs the learner to *abstract conceptualization* (AC), and finally ends with *active experimentation* (AE). At which point, the learning cycle goes back to concrete experiences etc. The model that uses RO (+ x), AE (- x), CE (+ y), and AC (- y) as axes, is known as the Kolb Learning Cycle. A learner in CE, deals with experiences preferring an intuitive approach to problems, compared to a more systematic and scientific approach. They relate well to people and are comfortable in real-life situations. A learner in RO, looks for meanings in situations through careful observations. They reflect on these meanings and seek to understand their implications. A learner in AC, focuses on logical understanding and concentrates on theoretical approaches to problems. Finally, a learner in AE, focuses on experimentation as opposed to reflective understanding. Further analysis gives us the following definitions.

- In the (AC,AE) quadrant is the *converger* whose dominant learning styles are abstract conceptualization and active experimentation. They do well in problem solving situations that have a single correct answer (physical sciences and engineering)

- In the (CE, RO) quadrant is the *diverger* whose dominant learning styles are concrete experience and reflective observation. They have the ability to generate alternate ideas (humanities and liberal arts).

- In the **(AC,RO)** quadrant is the *assimilator* whose dominant learning styles are abstract conceptualization and reflective observation. An assimilator will integrate observations and ideas into a theoretical framework (mathematics and basic sciences).

- In the (CE,AE) quadrant is the *accomodator* whose dominant learning styles are concrete experience and active experimentation. They adapt to changing circumstances and are action oriented (business and technical fields)

Because an ideal learner cycles through each of the learning modes and never actually settles at a single orientation, one should be cognizant of different learning styles when designing components of the learning labs.

APOS theory was developed to understand the ideas of reflective abstraction (AC,RO). APOS model for the construction of conceptual mathematical knowledge describes how Action becomes Process that can be viewed as Objects, as part of Schemas (Figure 1). Various instructional treatments such as corporative learning, ACE (Activity, Concept, Exercise), ICE (Instruction, Concept, Exercise), and programming languages (ISETL) have been successfully used in the past [1], [2], [3]. An **Action** is a change that an individual makes in a mathematical context requiring precise instructions to perform. A **Process** takes place when the individual begins to have control of the concept. An **Object** is constructed from a process when the individual becomes aware of the entire concept and understands that actions and processes can act on the concept. A **Schema** is when objects and processes from more than one area can be combined in more than one way.

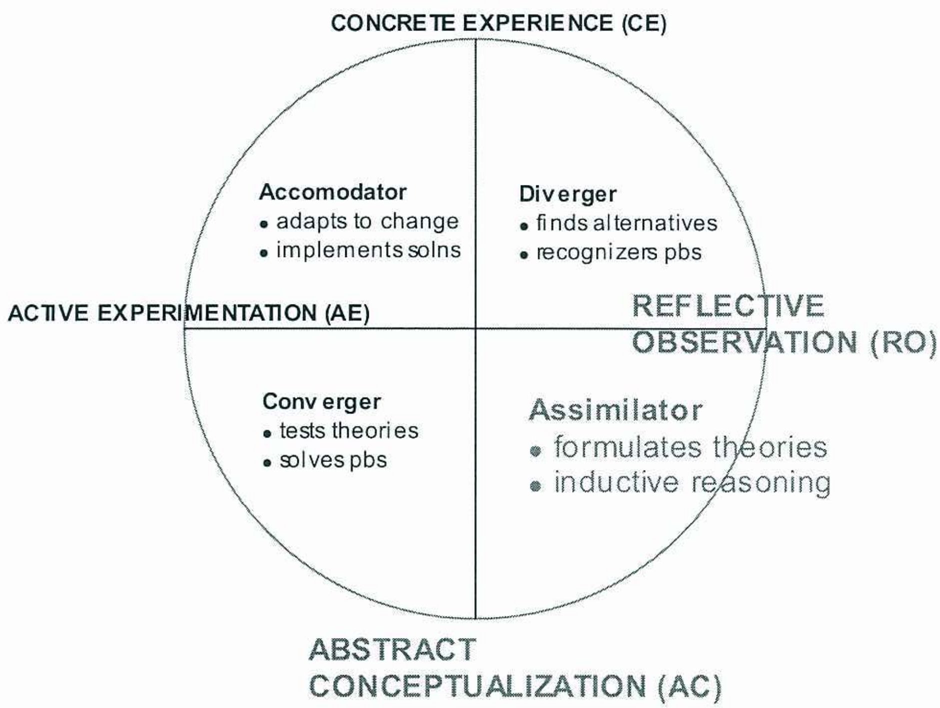

Figure 1: Kolb Learning Cycle and APOS quadrant

In the design of our learning labs we integrated the findings of Kolb Learning Cycle and APOS theory. Therefore, cooperative learning and the ACE cycle were our basis. We decided to employ a modified version of the ACE cycle, namely, the $\mathbf{ACEL}^2$ cycle. Here **A** stands for activity - the mini-video students use outside the class and before the lab, **C** stands for concept - discussed in the mini-video and in class prior to lab, **E** is for exercises contained within the lab, and $\mathbf{L}^2$ denotes the learning lab.

3 Learning Labs

The Learning Labs have the advantage being flexible enough to fit into existing topics in Linear Algebra. A handful of labs involve application topics from other disciplines. The courses are taught in a classroom equipped with laptop computers that can be arranged in a moments notice to simulate a laboratory environment. Specific labs are based on concepts, often extending ideas of matrix theory, and are designed using

Maple. There are several advantages to using Maple. A primary collection of these are listed below.

Interface: The Maple interface allows students to enter text and write mathematical statements on the same document. Features such as editing, examples, and the help menu allows students to experiment and explore during the lab.

Computation: Some of the more laborious procedures, such as row-reducing or finding an inverse of a given matrix, can be done efficiently, saving time for conceptual understanding. Instead of depending on the facilitator to dance around the table every few minutes to verify calculations, students can do their own checking and be confident knowing the mathematics they produce is correct.

Visualization: This is the most striking feature that enables students to move between different phases of APOS theory, weaving in and out of differing levels of understanding.

An outline of the topics covered in the Linear Algebra course is given in Table 2. The topics marked by ★ have an allied learning lab.

1. Linear Equations	**2. Matrix Algebra**	**3. Determinants**
★Systems of linear eqs ★Echelon forms Vector equations ★The matrix eq. $Ax = B$ ★Solutions of linear sy. ★Linear independence ★Linear transformations ★Matrix transformations	Matrix operations ★Inverse of a matrix ★Invertible matrices ★LU decomposition ★Applications	Intro. to dets. ★Prop. of dets. ★Cramer's rule
4. Vector Spaces	**5. Eigenvalues**	**6. Orthogonality**
Vector spaces & subsp. ★Null space & column sp. ★Linear indep. and bases ★Dimension of a vector sp. ★Rank	★Eigenvalues & eigenvec. ★Characteristic eq.	★Inner products Orthogonality ★Applications

Table 2: Topics of Learning Labs

3.1 Sample Labs

In this section we will explore one sample learning lab. A conscious choice was made to include a concept that extends beyond the usual topics covered in a typical Linear Algebra curriculum. The lab (on Circulant Matrices) can be categorized as one requiring only an Action level of manipulation. However, the resulting conclusion is non-trivial. Certain details of the lab are left out for brevity. Other labs such as the one on Schur's Theorem is on the Action and Process level of understanding. Schur's Theorem describes a fundamentally important fact for any matrix. The lab on unitary matrices brings together many ideas of matrix theory, exemplifying the deeper levels of learning connected with Objects and Schema in APOS theory. Sample labs can be accessed at http://www.personal.kent.edu/~akasturi/linalglabs.html. These interactive learning labs are easily adaptable with minor modifications to fit standard introductory Linear Algebra course.

Circulant Matrices

NAME: ______________ PARTNER: _____________ Linear Algebra - Lab 11

> *restart* : *with*(*plots*) : *with*(*linalg*) :

What are Circulant Matrices?

A matrix $A \in M_n(\mathbb{R})$ is called a circulant matrix if it has the form

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & \cdots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \cdots & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & \cdots & a_{n-3} & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_3 & a_4 & a_5 & \cdots & \cdots & a_1 & a_2 \\ a_2 & a_3 & a_4 & \cdots & \cdots & a_n & a_1 \end{bmatrix}.$$

Circulant matrices appear in problems in physics, image processing, probability, numerical analysis, and number theory. Most matrix-theoretic properties related to circulant matrices can be resolved concretely. This in itself places circulant matrices as objects to be studied for deep understanding. Our goal in this lab is to prove that two circulant matrices commute.

Rewriting Circulant Matrices

Our first goal is to understand how to rewrite circulant matrices using the powers of an elementary permutation matrix that we encountered in the course.

Let us denote the 3×3 permutation matrix (which is also circulant) as C.

$$> C := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Start with the general 3×3 circulant matrix B.

> $B := \begin{bmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{bmatrix}$

First let us compute powers of C.

> $C^2 \quad C^3$

Look carefully at the structure of C, C^2, C^3. Write the matrix B as a linear combination of C, C^2, C^3. Write your answer below.

Next let us do the same for a 4×4 matrix. Start with the standard 4×4 permutation matrix E and a general 4×4 circulant matrix F.

> $E := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad F := \begin{bmatrix} a & b & c & d \\ d & b & a & c \\ c & d & b & a \\ a & c & d & b \end{bmatrix}$

Compute powers of E.

> $E^2 \quad E^3 \quad E^4$

Write the matrix F as a linear combination of E, E^2, E^3, E^4. Write your answer below.

General Result

Lemma A matrix A can be written in the form $A = \sum_{k=0}^{n-1} a_{k+1} C^k$ iff A is circulant. Here C refers to the $n \times n$ permutation matrix.

Proof:

Circulant Matrices commute

Use the Lemma above to prove the following Proposition.

Proposition Any two circulant matrices commute.

Proof:

Example

Verify the Proposition using the matrices P, Q given below.

> $P := \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix} \quad Q := \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$

Useful Commands and References

MatrixMatrixMultiply

Davis, P., J., *Circulant Matrices*. John Wiley & Sons, Inc., 1976

4 Conclusions

Pedagogy based on Vygotskian theory [8] places the learning of mathematics as a conceptual endeavor. The Learning Labs for Linear Algebra designed in Maple capture the essential ingredients of experiential learning and APOS theory. Teaching conceptual knowledge first leads to the acquisition of procedural skill. However, conceptual and procedural skill are to be treated as part of a single cognitive schema. We have introduced the $\mathbf{ACEL}^2$ cycle to justify the use of learning labs to advance students to a higher level of cognition. Measurement and comparison of a Conceptual Performance Index (CPI) and Skill Performance Index (SPI), as done in [4], will be investigated in the future for the redesigned Linear Algebra courses.

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