DIRECT ESTIMATION OF THE AREA OF A LAKE AND 'GROUND TRUTH' BY GPS

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I. Introduction

We describe an engaging exercise in geometry that we propose in a general mathematics course, namely the measurement of the surface area of an irregular surface, such as a lake. The project is not trivial, but not too difficult either, and has the benefit of active participation of the students in several aspects, and thus will help greatly in the teaching and learning of mathematics. We propose to estimate the area of an amorphous lake by direct measurements of lengths and widths at various sections of the lake and summing inscribed trapezoids to represent the area.

"Lake Lion" is an amorphous lake sprawled in the middle of the campus at Florida Memorial University. Its amoeba-like shape is a perfect challenge for an exercise such as this one for it tests one's ability to 'cut' around curves by straight line segments. But a result can be obtained after a judicious choice of the size of elemental trapezoids.

Florida Memorial University is a coeducational college institution in Miami, the only Historically Black College and University in South Florida. Founded in 1879 in Live Oaks, it has a long history of name and site changes. Its campus in St Augustine had 110 acres of area. It became a 4-year baccalaurette institution in 1941, moved to its present-day campus in Miami in 1968, and became a university in 2004. At present, it has offerings of 41 undergraduate and 4 graduate degree programs, with a student body that is 2000 strong. In the middle of this 44-acre campus is a large man-made lake that is so

irregular in shape that it would present a challenging but not insuperable problem that would engage a mathematically curious general education class. We will call this lake, "Lake Lion," after our school mascot. Unfortunately, our public archives do not tell us the surface area of this lake with which to compare our estimates, so we will have to come up with our own standard value.

II. The project

Figure 1 shows a panoramic view of the lake looking south. Other views of the lake are available and may be seen by request from the author. The singular advantage of this man-made lake is that its boundary is lined by concrete so that the lake is constant in area and perimeter and saves us the trouble of allowing for variable edge effects. Our most interesting view of the lake is via satellite from *Google Earth*, Figure 2. It will be noted that the coordinates are $25^0 55'$ N, $80^0 16'$ W, with an eye altitude of 176 m. This is one of several views of the lake that we used to calculate the surface area. On paper, we drew lines across the face of the lake, and estimate its area by inscribed trapezoids. Altogether we counted 24 trapezoids to sufficiently cover the lake's surface (Figure 3). From calculus, we know that a sufficiently large number of rectangles, or trapezoids, or parabolas, can estimate the area bounded by a curve to any predetermined accuracy. We chose trapezoids because their shape approximates better the shape of the lake edges; we also chose inscribed, rather than circumscribed, trapezoids for better accuracy. For real-life values of the area, we scaled the picture by measuring the dimensions of a known existing rectangular bar near the north end of the lake.

It will also be noted that the satellite view of the lake furnishes two extra data: the perimeter and the area of the lake. This can be obtained directly from the satellite view by moving the cursor around the lake and digitizing its outline. The values of the perimeter and area are directly shown on the picture itself. This satellite datum serves as our standard value for the surface area.

A third method to estimate the surface area is by the use of age-old planimetry. The planimeter, if available, could give us the area enclosed by the curve as drawn by the device. Unfortunately, we did not have this device. So we used the next best thing. We measured the weight of the paper cutouts of the lake in an analytical balance, and compared this weight to the total weight of a known area at the site. Scaling the ratio results in the actual area of the lake to real-life values.

III. Analysis of the results

A. Estimate by direct quadrature of inscribed trapezoids

In Figure 3, we show the inscribed trapezoids covering the lake surface. There are altogether 24 slices, the two end slices being slightly thinner than the rest, and slice 13 being a composite of a trapezoid and two rectangles. The calculation of each elemental slice is straightforward, and the results are shown in Figure 4. Using a conversion factor to actual size, the lake area is estimated to be 76,522.74 square feet.

B. 'Ground truth' by GPS

To test the accuracy of our rough estimate, such as just performed, requires a reference value to serve as standard. Unfortunately, there is no such reference value extant in the school records. We used the value provided by the GPS satellites of *Google Earth*; this serves as our 'ground truth'. Google Earth is available free on a trial period. It may be used to determine locations and distances between towns, and estimates of surface area of indicated regions on the Earth. More than occasional use of this facility requires a contract with Google Earth. More recently Google Earth also provides the topography of the sea bottom, and even star maps, although it may be suspected that these data are non-satellite-based and may have been derived elsewhere.

What concerns us here are the data the satellites yield for the surface area of the digitized region. It turns out that this value is dependent on how closely-spaced you digitize the boundaries of the region, though the variation is very small. For our case, the area of the lake is given as 85,419.47 square feet, about 1.92 acres. Our estimate by trapezoids is smaller by 8.3% error from the standard value.

C. Density measurements of lake cutouts

Further confirmation of our estimates is done using planimetry. Pictures of the lake taken from satellites (Google Earth) are cut out (Figure 4) and weighed on an analytical balance, and compared against the weight of the whole paper sheet of known area. Our results yield 81,862.30 square feet as the area of the lake. The figures obtained compared favorably, with a standard error of 1.9%. It may be remarked that the satellite value for the surface area is really just the result of a sophisticated planimeter. The value obtained by satellite requires digitization of the lake boundary, that is, defining the outlines of the lake as a planimeter device would, and using some undetermined algorithm to calculate the area. The satellite estimate is just a modern-day sophisticated planimeter.

IV. Future work

Future work for this project will be geared towards improving the estimates obtained. For one, the number of trapezoidal slices can be increased, maybe even doubled. It may be expected that the results will be improved as the number of elemental slices is increased. Second, large figures, like a "+" sign, can be planted near the lake to serve as reference for the satellite scans to determine the aspect ratio between the x- and y-coordinates. Third, the planimetry estimates can be improved by either getting an actual planimeter device to carry out the measurement, or getting better planimeter paper with which to weigh the cutouts of the lake. Also, our satellite views showed that the perimeter of the lake can also be obtained as a bonus parameter. This parameter can be included in our future projects.

V. Conclusion

The project presented here is planned to be proposed to start in the fall term in one of the general math education courses. This should be a welcome addition to the teaching and learning of general interest mathematics courses because it would actively engage everyone in class. Some students can take measurements of the lake parameters, some students can do research on the Google satellites, other students can take planimeter readings, and others can take photographs of varying lake views. And everybody will be involved in organizing and presenting the results.

Preliminary attempts at measuring the surface area of an irregularly-shaped body, such as a lake show that the estimates are promising. Three different methods of measurement – by trapezoidal slices, by satellite, and by planimetry -- were made and their results compared favorably. The best estimate gave a value of 1.9 acres for the surface area of an amorphous lake.

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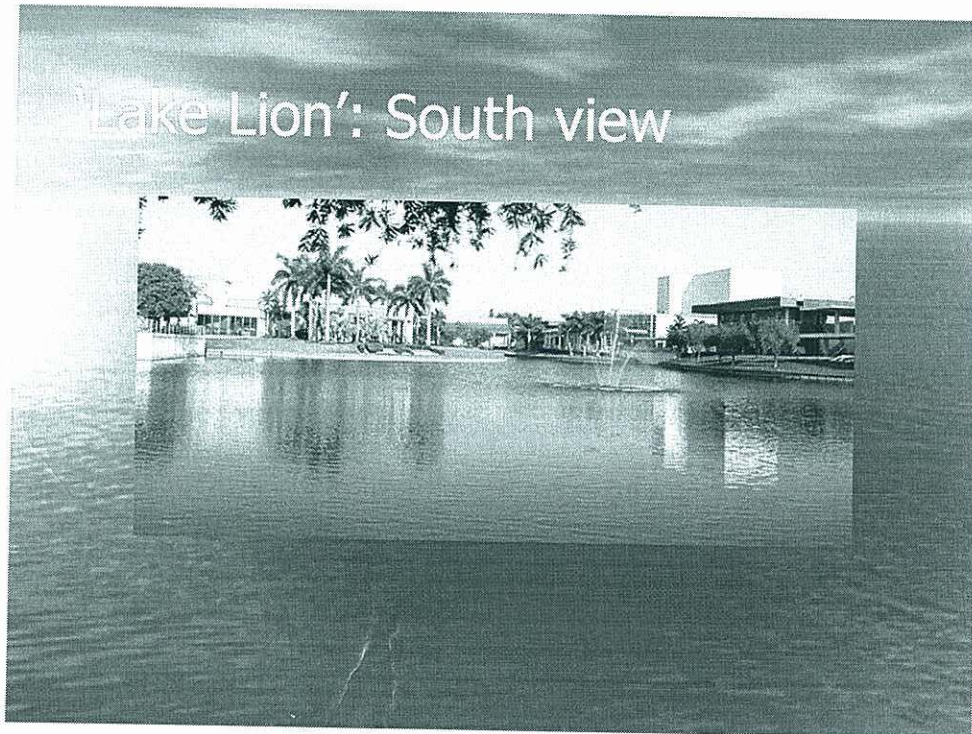
Figure 1: 'Lake Lion': South view

'Lake Lion': South view

Figure 2: Google Earth: Satellite view
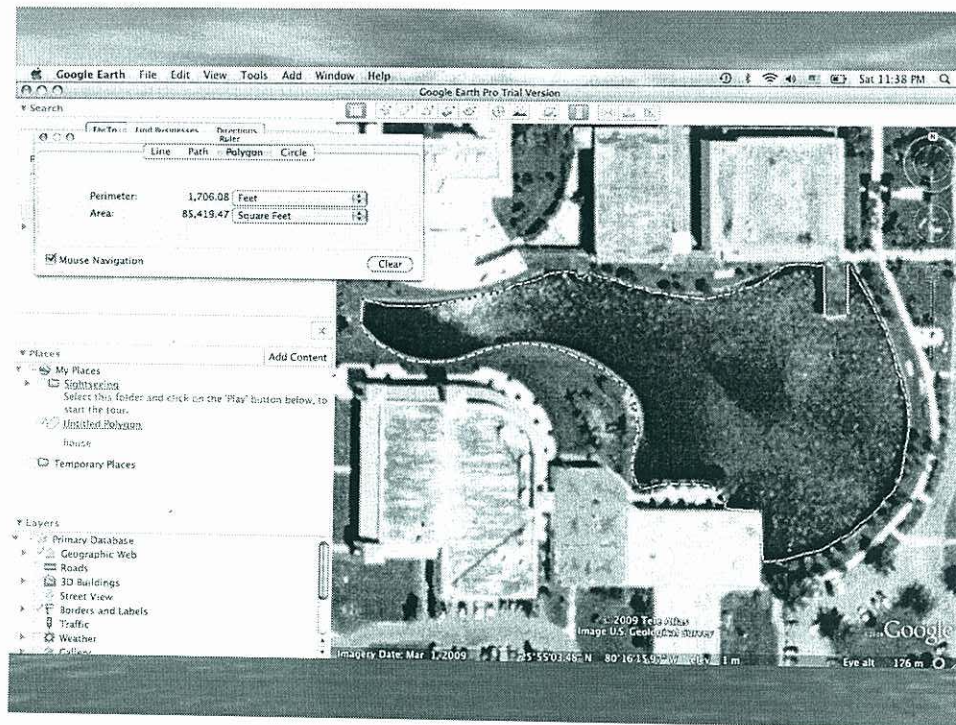
Figure 3: Area by trapezoids

Area by trapezoids

$= A1 + A2 + \ldots + A24;$ $A1 = \frac{1}{2}(a+b)h$ $A24 = \frac{1}{2}(bh)$

$A1 = \frac{1}{2}(5+7)3$ $A9 = \frac{1}{2}(14+16)5$ $A17 = \frac{1}{2}(45+47)5$
$A2 = \frac{1}{2}(7+9)5$ $A10 = \frac{1}{2}(16+18)5$ $A18 = \frac{1}{2}(47+48)5$
$A3 = \frac{1}{2}(9+11)5$ $A11 = \frac{1}{2}(18+19)5$ $A19 = \frac{1}{2}(61+62)5$
$A4 = \frac{1}{2}(11+13)5$ $A12 = \frac{1}{2}(19+20)5$ $A20 = \frac{1}{2}(62+60)$
$A5 = \frac{1}{2}(13+12)5$ $A13 = 21(5)+8(5)+\frac{1}{2}(5+3)12$ $A21 = \frac{1}{2}(50+47)$
$A6 = \frac{1}{2}(12+11)5$ $A14 = \frac{1}{2}(41+39)5$ $A22 = \frac{1}{2}(47+44)5$
$A7 = \frac{1}{2}(12+11)5$ $A15 = \frac{1}{2}(39+39)5$ $A23 = \frac{1}{2}(44+29)5$
$A8 = \frac{1}{2}(12+14)5$ $A16 = \frac{1}{2}(39+41)5$ $A24 = \frac{1}{2}(29)2.5$

$A = 3348.5 \text{ mm}^2;$ * W = 5 mm = 20' 5.25"; L = 11 mm = 61.5'
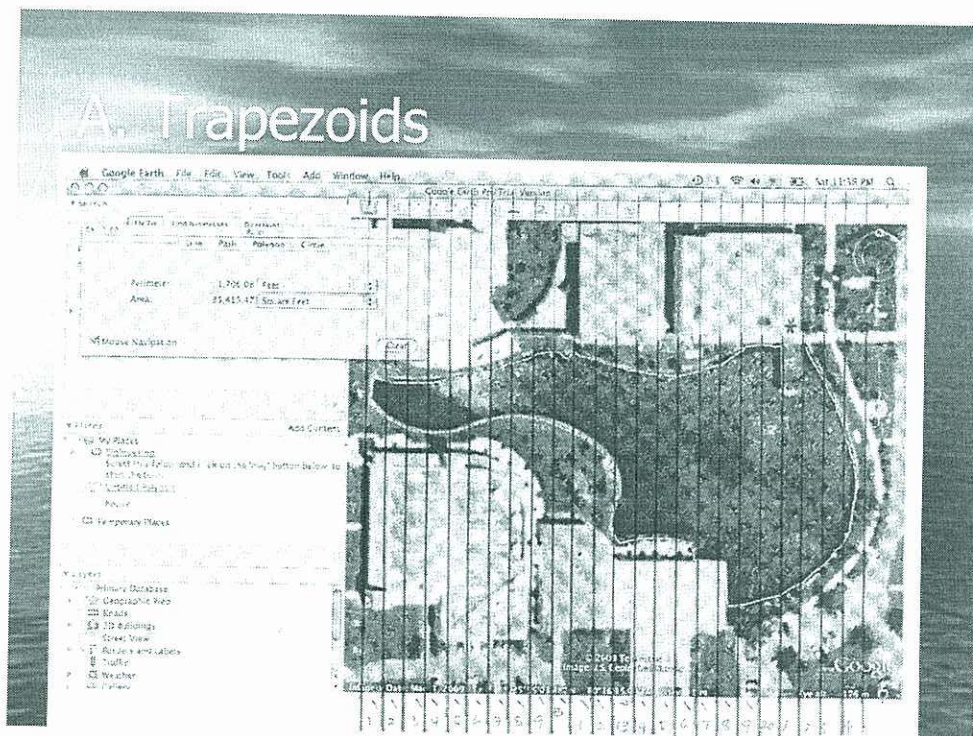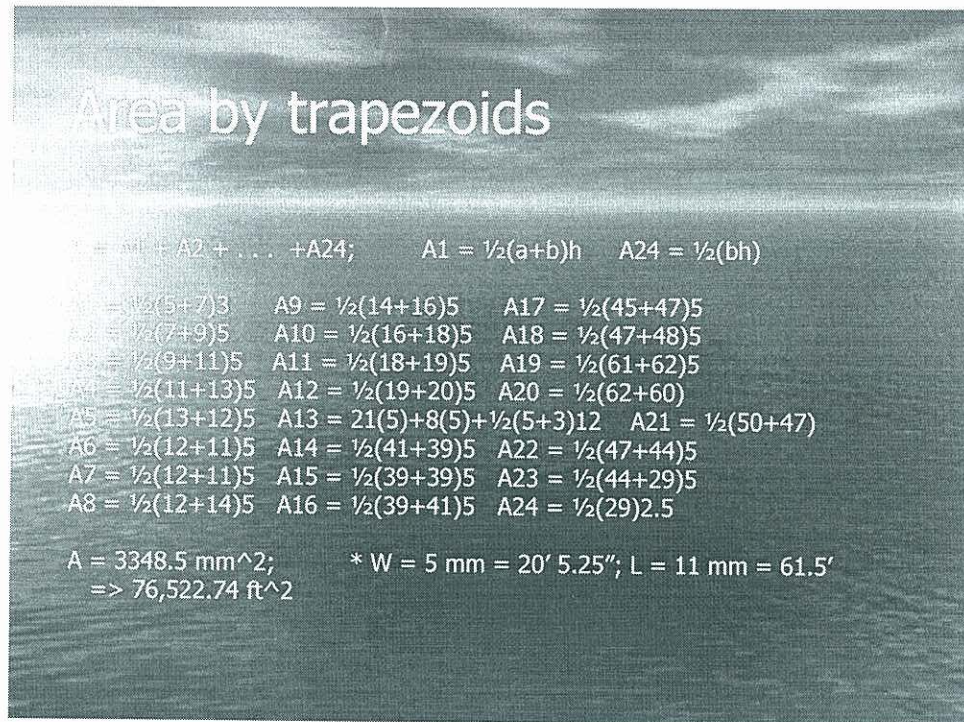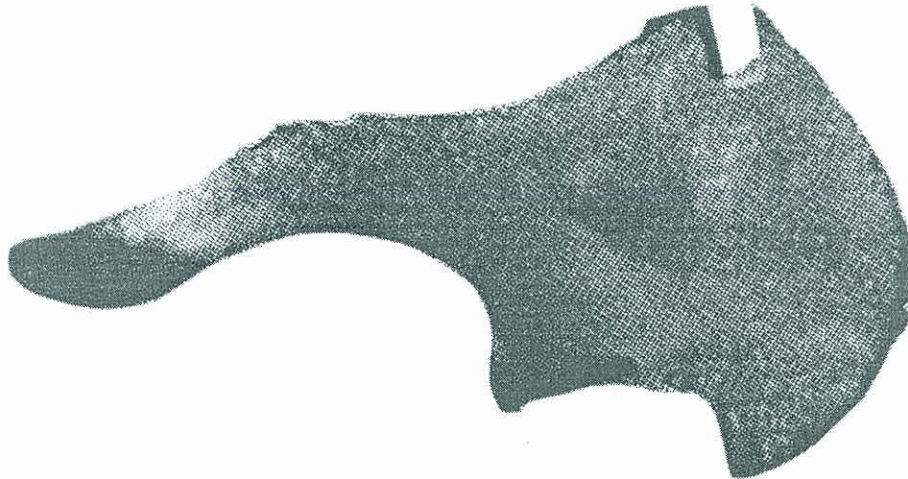$\Rightarrow 76{,}522.74 \text{ ft}^2$

Figure 4: Area by planimetry

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Planimetry

Analytical balance: (c/o Dr Mike Elliott, Chemistry)
Cutout Area = 276.20 mg
Paper Area = 4525.65 mg

Paper: 8.5 x 11 in^2 = 215 x 273 mm^2
          = 58695 mm^2
Cutout: (276.20/4525.65)(58695) = 3582.15
     mm^2 => 81,862.30 ft^2
% error = 1.9%

References:

1.*Google Earth Satellites*, Internet.
2. Thomas, G, 2005. *Calculus.* Boston: Addison Wesley.