THE ROLE PLAYED BY THE TI-89 CAS IN RESOLVING DIVISIBILITY QUESTIONS

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Abstract: Tests for the divisibility of an integer by 3, 5, 7, and 11 are well known. We focus on the role played by CAS technology in resolving integer divisibility by other primes including 37, 41, 73, and 101. In addition, CAS technology together with modular arithmetic are employed to secure prime divisors in the sequence 11, 101, 1001, 10001,… first studied in 1854. We begin with some basic divisibility ideas that the reader is undoubtedly familiar with.

In order to test for divisibility of an integer N by the integers 3, 5, 7, and 11, one can resort to the following tests where in canonical form
$N = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + a_{n-2} \cdot 10^{n-2} + a_{n-3} \cdot 10^{n-3} + ... + a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$.

N is divisible by 3 if and only if

$$N = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + a_{n-2} \cdot 10^{n-2} + a_{n-3} \cdot 10^{n-3} + ... + a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \equiv a_0 + a_1 + a_2 + ...(\text{mod}\,3).$$

This follows since each of the following is true using basic properties of congruences:

$1 \equiv 1(\bmod 3) \Rightarrow a_0 \equiv a_0(\bmod 3).$

$10 \equiv 1(\bmod 3) \Rightarrow a_1 \cdot 10 \equiv a_1 \cdot 1(\bmod 3) = a_1(\bmod 3).$

$10^2 \equiv 1^2(\bmod 3) \Rightarrow a_2 \cdot 10^2 \equiv a_2 \cdot 1(\bmod 3) = a_2(\bmod 3).$

$10^3 \equiv 1^3(\bmod 3) \Rightarrow a_3 \cdot 10^3 \equiv a_3 \cdot 1(\bmod 3) = a_3(\bmod 3).$

....

$10^n \equiv 1^n(\bmod 3) \Rightarrow a_n \cdot 10^n \equiv a_n \cdot 1(\bmod 3) = a_n(\bmod 3).$

Recall that $a \equiv b(\bmod n) \Leftrightarrow n \mid (a-b);\ a,b,n \in \mathbb{Z}$ *and* $n \geq 2$. In similar fashion, the following are true:

N is divisible by 5 if and only if

$N = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + a_{n-2} \cdot 10^{n-2} + a_{n-3} \cdot 10^{n-3} + \ldots + a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \equiv$
$a_0(\bmod 5).$

N is divisible by 7 if and only if

$N = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + a_{n-2} \cdot 10^{n-2} + a_{n-3} \cdot 10^{n-3} + \ldots + a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \equiv$
$a_0 + 3 \cdot a_1 + 2 \cdot a_2 - 1 \cdot a_3 - 3 \cdot a_4 - 2 \cdot a_4 + 1 \cdot a_5 + \ldots(\bmod 7).$

N is divisible by 11 if and only if

$N = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + a_{n-2} \cdot 10^{n-2} + a_{n-3} \cdot 10^{n-3} + \ldots + a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \equiv$
$a_0 - a_1 + a_2 - a_3 + \ldots(\bmod 11).$

The TI-89/VOYAGE 200 has the capabilities using the mod or remainder functions to generate the sequence of repeating multipliers needed to verify these tests. Consider the following illustrated in FIGURES 1-5 below:

FIGURE 1 (SEQUENCE MODE)

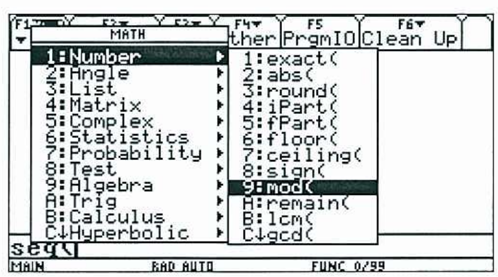

FIGURE 2 (THE MOD COMMAND)

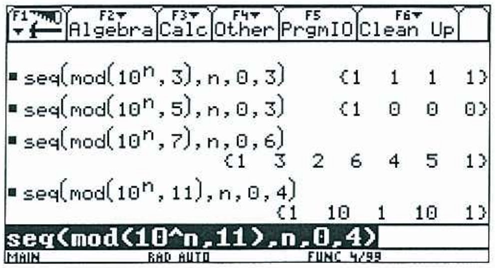

FIGURE 3 (sequence of remainders)

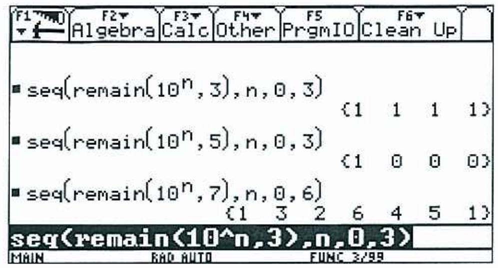

FIGURE 4 (sequence of remainders)

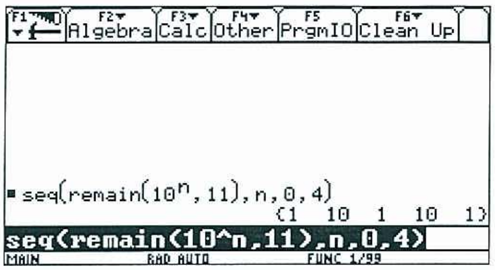

FIGURE 5 (sequence of remainders when 10^n is divided by 11 for $n = 0-4$ inclusive).

Observe in FIGURE 5 that the remainders obtained in the test for divisibility by 11 for the integers 1, 10, 100, 1000, and 10000 proceeding from right to left are respectively 1, 10, 1, 10, and 1 instead of 1, -1, 1, -1, and 1 as we are used to seeing in the alternating sum of the digits test for divisibility of an integer by 11. The calculator chooses to use only remainders that are 0 or positive. Of course, -1 and 10 have the same remainder of 10 when divided by 11 so that both remainder schemes are correct $\left(-1 \equiv 10(\text{mod}11)\right)$.

One could likewise resort to a TABLE where the sequences are entered in FUNCTION MODE and the outputs are the remainders. This will only accurately work through $x = 14$. Thereafter the outputs default to 0. See FIGURES 6-9 below:

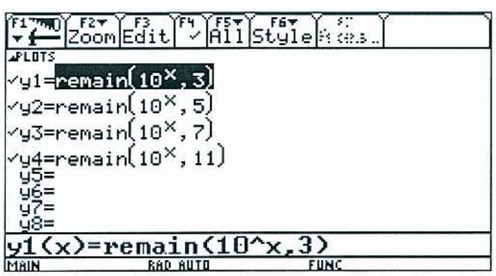

FIGURE 6 (Y = EDITOR)

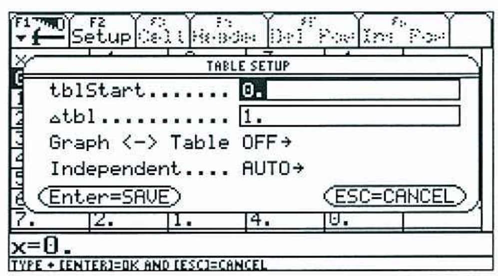

FIGURE 7 (TABLE SETUP)

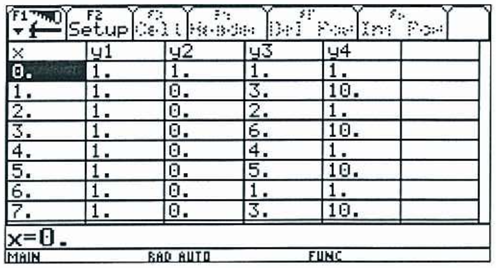

FIGURE 8 (sequence of remainders)

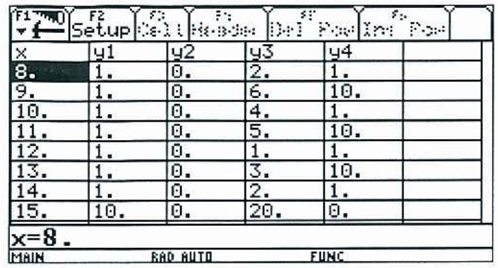

FIGURE 9 (sequence of remainders)

While the outputs are incorrect in FIGURE 9 for all of the above functions save the second when $x=15$, Wolfram MATHEMATICA 7 generates correct results below:

```
Table[Mod[10^n,3],{n,1,50}]
```

```
{1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1}
```

```
Table[Mod[10^n,5],{n,1,50}]
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{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}

Table[Mod[10^n, 7], {n, 1,50}]

{3,2,6,4,5,1,3,2,6,4,5,1,3,2,6,4,5,1,3,2,6,4,5,1,3,2,6,4,5,1,3,2,6,4,5,1,3,2,6,4,5,1,3,2,6,4,5,1,3,2}

Table[Mod[10^n, 11], {n, 1,30}]

{10,1,10,1,10,1,10,1,10,1,10,1,10,1,10,1,10,1,10,1,10,1,10,1,10,1,10,1,10,1}

Our next goal is to discuss divisibility tests for any integer by the integers 37, 41, 73, and 101. We resort to the calculator as illustrated in FIGURES 10-11 below:

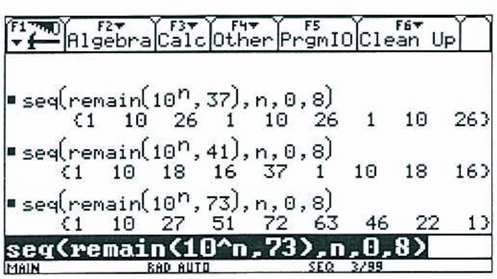

FIGURE 10 (sequence of remainders)

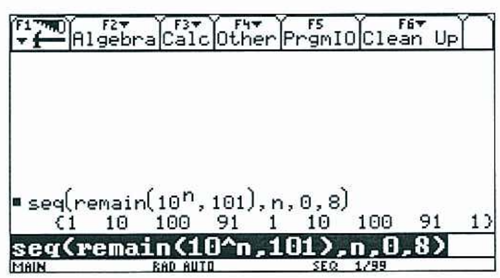

FIGURE 11 (sequence of remainders)

The sequence of multipliers going from right to left that repeats is 1, 10, -11, 1, 10, -11, 1,... in the test for divisibility by 37. We note that $26 \equiv -11 \pmod{37}$.

In the test for divisibility of an integer by 41, the sequence of multipliers from right to left is 1, 10, 18, 16, -4, 1, 10, 18, 16, -4, 1,.... Again observe that $37 \equiv -4 \pmod{41}$.

In the test for divisibility of an integer by 73, the sequence of multipliers from right to left is 1, 10, 27, -22, -5, -10, -27, 22,1,.... Note that the following congruences are valid:

$-22 \equiv 51 \pmod{73}$.
$-5 \equiv 68 \pmod{73}$.
$-10 \equiv 63 \pmod{73}$.
$-27 \equiv 46 \pmod{73}$.

Finally in the test for divisibility of an integer by 101, the sequence of multipliers from right to left is 1, 10, -1, -10 1, 10, - 1, -10 , 1,.... It is immediate that the following congruences are valid:

$100 \equiv -1 \pmod{101}$.
$91 \equiv -10 \pmod{101}$.

Let us prove the last of the congruences above:

$1 \equiv 1 \pmod{101} \Rightarrow a_0 \equiv a_0 \pmod{101}$.
$10 \equiv 10 \pmod{101} \Rightarrow a_1 \cdot 10 \equiv 10 \cdot a_1 \pmod{101}$.
$10^2 \equiv -1 \pmod{101} \Rightarrow a_2 \cdot 10^2 \equiv -a_2 \pmod{101}$.
$10^3 \equiv -10 \pmod{101} \Rightarrow a_3 \cdot 10^3 \equiv -10 \cdot a_3 \pmod{101}$.
$10^4 \equiv (-1)^2 = 1 \pmod{101} \Rightarrow a_4 \cdot 10^4 \equiv a_4 \pmod{101}$.

Thus it follows that

$$N = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + a_{n-2} \cdot 10^{n-2} + a_{n-3} \cdot 10^{n-3} + \ldots + a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0 \equiv$$
$$a_0 + 10 \cdot a_1 - a_2 - 10 \cdot a_3 + a_4 + \ldots \pmod{101}.$$

The centerpiece illustrating a neat application of congruences in this paper will focus on the integer sequence 11, 101, 1001, 10001, 1000001,... which can elegantly be represented in closed form as $f(n) = 10^n + 1$. While 11 and 101 are prime, are there any other primes in this sequence? It should be fairly evident that any sequence with an even number of 0's corresponding to an odd value of n is divisible by 11, and since the integer is greater than 11, not prime. In such an instance, the units digit is multiplied by 1 and the leftmost digit is multiplied by -1 so that the sum is 0 which is divisible by 11 $(0 = 11 \cdot 0)$. Let us factor the first twenty integers of this form with the aid of the TI-89. The Table is provided in FIGURES 12-15:

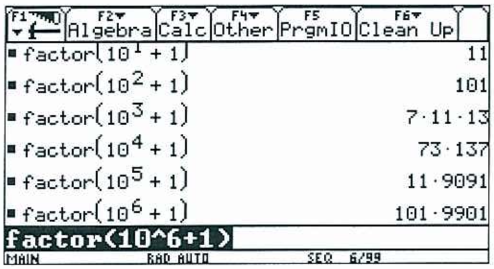

FIGURE 12 (factoring $10^n + 1$, $n = 1 - 6$.)

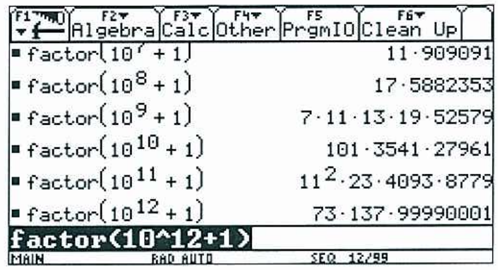

FIGURE 13 (factoring $10^n + 1$, $n = 7 - 12$.)

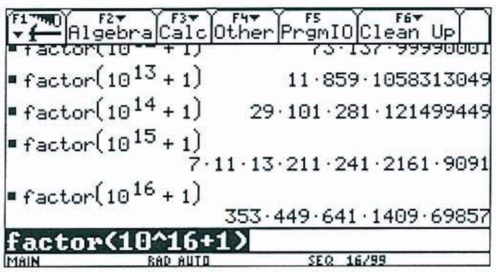

FIGURE 14 (factoring $10^n + 1$, $n = 13 - 16$.)

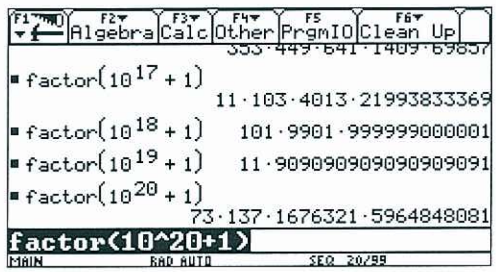

FIGURE 15 (factoring $10^n + 1$, $n = 17 - 20$.)

Our next goal will provide the entry point that a prime enters this integer sequence:

Prime:	First few values of n:
3	None
5	None
7	3, 9, 15,…
11	All odd values: 1, 3, 5, 7, 9,…
13	3, 9, 15,…
17	3, 9, 15,…
19	9, 27, 45,…

23	11, 33, 55, 77,…
29	14, 42, 70,…
31	None
37	None
41	None
43	None
47	23, 69, 115, 161,…
53	None
59	29, 87, 145,…
61	30, 90, 150,…
67	None
71	None
73	4, 12, 20, 28,…
79	None
83	None
89	22, 66, 110,…
97	48, 144, 240,…
101	2, 6, 10, 14,…

We notice that not all primes enter the sequence. For example, the primes 37, 41, and 43 do not enter the sequence. Let us see why with the aid of our CAS technology utilizing FIGURES 16-18 below:

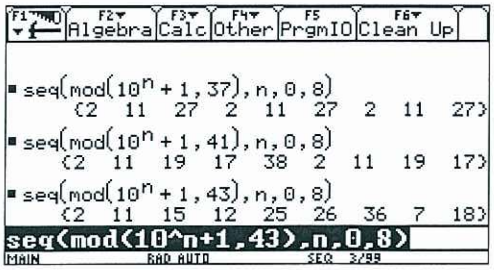

FIGURE 16 (sequence of remainders)

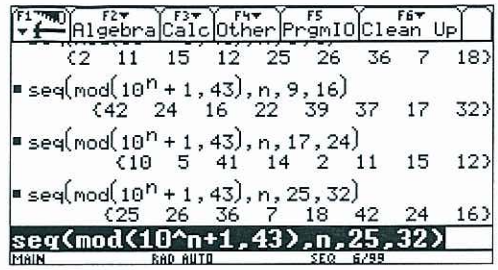

FIGURE 17 (sequence of remainders)

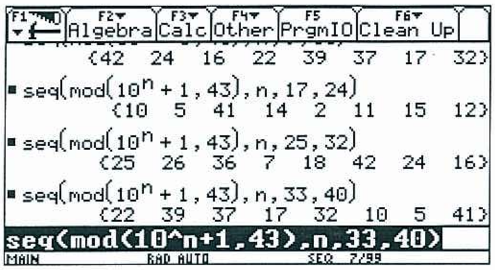

FIGURE 18 (sequence of remainders when $10^n + 1$ is divided by 43, $n = 17 - 40$ inclusive.

On the other hand, the primes 73 and 101 enter the sequence as claimed (notice the 0 in such instances as an output) in FIGURES 19-21 below:

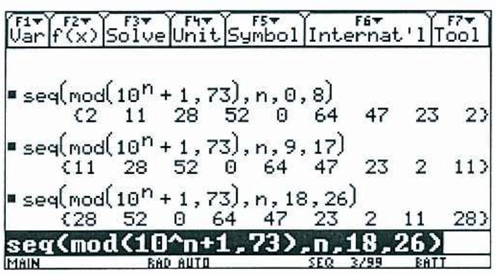

FIGURE 19 (sequence of remainders)

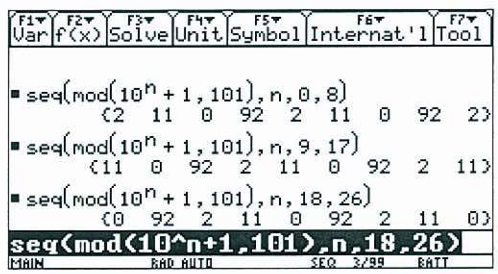

FIGURE 20 (sequence of remainders)

Conclusion: CAS technology enables students to discover mathematics and form conjectures. Modular arithmetic is a neat vehicle for exploring the concept of divisibility and applying it in a variety of settings to various integer sequences. The reader is invited to partake of these ideas further and to possibly resolve the question as to whether there exist additional primes in the aforementioned sequence other than 11 and 101. This problem was first posed by Reverend James Booth in 1854.

Reference:

[1]. Thomas J. Osler and John F. Kennedy, The Integers of James Booth, Mathematical Spectrum, 39 (2006/2007), pp. 71-72.