

Discrete Dynamical Systems with the TI-Nspire CAS

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Discrete Dynamical Systems (DDS) are powerful, intuitive modeling tools for describing and predicting changing situations. Because my workshop was mostly about how to use the TI-Nspire in looking at DDS, this paper will be about a few insights attained about the technology but mostly about examples DDS problems I have used.

Model a prescribed drug dosage.

One time drug: Novocain (Procainamide Hydrochloride) injected as an anesthetic for minor surgical and dental procedures is eliminated from the body primarily by the kidneys. Loosely speaking, during any 1-hour period, the kidneys take a fixed percentage of the blood and remove medicine from the blood. Let's assume the kidneys purify 1/5 of the blood every one-hour period.

Let $u(n)$ = # of mg of prescribed drug in our system after n (one-hour) periods.

Then $u(n) = u(n-1) - .20u(n-1) = .80u(n-1)$, $n = 1, 2, 3, \dots$

Let's assume we are given 500 mg of the drug in period 0. So $u(0) = 500$.

This model is a discrete dynamical system, DDS for short. Later we will iterate and graph the function u to see what happens over a long period of time. In the past I have demonstrated how the TI-83/84 or EXCEL can be used. This paper will focus on how the TI-Nspire CAS can be used. Some background might help the mathematics teacher. Using a DDS involves modeling with the paradigm:

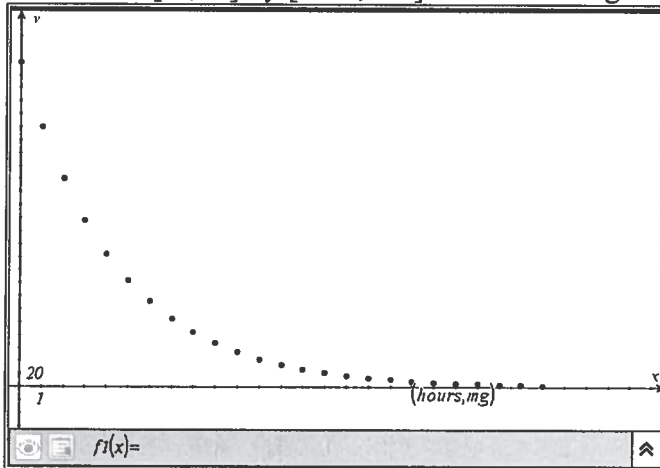
$$\text{Future} = \text{Present} + \text{Change}.$$

A DDS is a discrete function that can be used to model many situations, such as mortgage of a home, car financing, investment or financial alternatives, prescribed drug dosages, population dynamics such as predator-prey problems, competitive hunter problems, and genetics. Many of these examples are covered in this article, but there are many more. Discrete dynamical system models are some of the more robust models known. A DDS might sound familiar from calculus which is the study of change. In calculus our paradigm is changed to $\boxed{\text{change} = \text{future value} - \text{present value}}$ which is known as a Difference Equation.

Let's return to our example. We can build the solution on the TI-Nspire CAS or non CAS or any other calculator with an ANS key. Just put in the initial value and hit **enter**, then ***.8** and **enter**. The value displayed on the screen is $u(1)$. We write on our paper $u(1)=400$. Next we hit **enter** and write on our paper $u(2)=320$. And so forth until you have achieved your desired long term behavior.

Now let's use the TI-Nspire spreadsheet. Hit the **HOME** (the little house) key and select **3:Lists & Spreadsheets**. Hit the cursor key to move up two cells to provide the title **hours** and move down two cells to A1. Hit the **menu** key and select **3:Data** and **1:Generate Sequence**. Type in $u(n)=u(n-1)+1$ and **enter**. For Initial Terms: type in **0**. For Max No. Terms: type in **25** (for the 24 hours in a day plus 0). You can leave the ceiling value blank. Move to and hit **OK**. Move to the right one cell, and move up two cells. Here type in the title **mg** and move down two cells. Hit the **MENU** key and select **3:Data** and **1:Generate Sequence**. Type in $u(n)=.8*u(n-1)$ and **enter**. For Initial Terms: type in **500**. For Max No. Terms: type in **25** (for the 24 hours in a day plus 0). For Ceiling Value: type in **505** (something greater than 500). Hit **OK**. Notice that the values in the **mg** column match the values from your iteration and that the mg of Novocaine are headed slowly to zero.

Next we will graph our sequences. Hit **home** and select **2:Graphs & Geometry**. Next hit **menu** and select **3:GraphType**, **4:Scatter Plot**. For the x choose **hours** and for y choose **mg**. Now hit **menu** and select **4:Window** and **1:Window Settings**. Set up a window of $[-5,25]$ by $[-150,550]$. You should get a graph similar to that below.



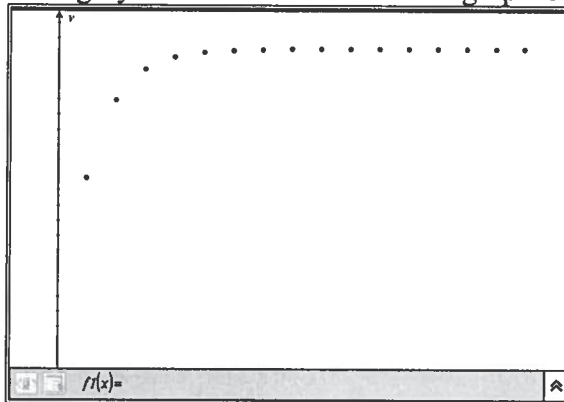
More Than One Time Drug Dosage

Now, let's consider a drug taken more often than once. *CIPRO* is a drug for combating many infections, including anthrax. Let's assume that during a one-hour period that our kidneys purify 1/4 of this drug from the blood. Let's assume that the dosage is 16 mg each time period. Let's see what happens between each dosage.

Write the mathematical model that represents this system. Our model is $d(n) = .75 d(n-1)$, $d(0)=16$ mg, $n=0,1,2,3,\dots$ (in one-hr periods). Let's assume that to be effective, you must have at least 6.75 mg of the medicine in your blood. How often should you take the medicine?

Now, let's remodel assuming that every 4-hour period we take 16 mg of the drug and that we don't have any in our system when we begin, $d(0)=0$. Let's assume that in a 4-hour period the kidney's only purify 60% of the drug. The model is $d(n)=d(n-1)-.6*d(n-1) + 16$ or $d(n) = .40 * d(n-1) + 16$ for $n=0,1,2,3,\dots$ where n now represents 4-hour periods.

In the course of preparing this workshop, I learned a different method for “generating the sequence.” As in the previous example, we start a new spreadsheet. Let’s set up our first column of 4-hour periods. I found out that you cannot start a variable with a number, so I titled the column *four*. In cell *a1* I put 0. Then I moved down to cell *a2* and input the formula $=a1+1$ and hit **enter**. Next I moved back up to cell *a2* and hit **ctrl** and “click” (where the hand is). This causes a dotted border to appear around cell *a2*. Now we can drag this formula down as far as we want (let’s say 20) by pressing the down arrow. When you see what you want press **enter**. You should see the numbers 1 to 19 or 20 displayed in the column. Next press the up key, which should bring you back to cell *a1*. Next move the cursor over to column *b*, title it *mg2* or *Cipro*. Move to *b1* and type in 0. Press **enter** or the down arrow to cell *b2*. Here type in the formula $=0.4*b1+16$ and press **enter**. Now move back up to *b2* and “grab and drag” this formula (by **ctrl** “click” and the down arrow) down to cell *b21*. Now you have the iterations you need. What happens to the Cipro over eighty hours? Now let’s make graph of the data. Your graph should



look similar to

Describe what you see from the graphical output. This shows a stable equilibrium value. The definition of an equilibrium value is when $d(n)=d(n-1)$, $INPUT=OUTPUT$, and there is no change. This equilibrium value is _____mg of this drug.

In the workshop because the TI-Nspire is relatively new technology, we were nearly out of time at this point. I covered one more example of pollution in the Great Lakes of Erie and Ontario, which is a system of DDS, and talked a little about the Competitive Hunter Problem at the end. Below are listed other examples I had planned. In addition, I had planned to cover a Deer Population project and a project I made out of Jim Sandefur’s (See Resource at the end of the article) Embezzler Problem.

Car Finance Example

You want to buy a \$20,000 new car and you can afford a monthly payment of only \$400 per month. As you shop around you find a dealer that will give you \$1,500 cash back if used as a down payment and 6.9% per year financing compounded monthly. Can you get your new car?

Let $a(n)$ = the amount that you owe the financing company after n months

$$a(n)=a(n-1)+(.069/12) a(n-1) - 400$$

$$a(n) = (1 + .069/12) a(n-1) - 400$$

How can we solve this DDS? Well, one method is by numerical iteration. We know that $a(0)=\$18,500$. Go ahead and work through this problem.

Nonlinear Discrete Dynamical Systems

In this section we build nonlinear discrete dynamical systems to describe the change in behavior of the quantities we study. To remind us let's define a nonlinear DDS-- If the function of $a(n)$ involves powers of $a(n)$ (like $a^2(n)$), or a functional relationship (like $a(n)/a(n-1)$), we will say that the discrete dynamical system is **nonlinear**.

Spread of a Contagious Disease

There are 1000 students in a college dormitory and some students have been diagnosed with meningitis, a highly contagious disease. The health center wants to build a model to determine how fast the disease will spread.

Problem: Predict the number of students affected with meningitis as a function of time. Let $m(n)$ be the number of students affected with meningitis after n days. We assume all students are susceptible to the disease. The possible interactions of infected and susceptible students are proportional to their product (as an interaction term).

The model is $m(n) - m(n-1) = k m(n-1) (1000-m(n-1))$ or $m(n) = m(n-1) + k m(n-1) (1000-m(n-1))$. It is found that two students returned from spring break with meningitis. The rate of spreading per day is characterized by $k=0.0025$. It is assumed that a vaccine can be in place and students vaccinated within 1-2 weeks.

The results clearly show that most students will be affected within 2 weeks. Since only about 10% will be affected within one week, every effort must be made to get the vaccination at the school and get the students vaccinated within one week.

Chaos Example. Euler's Method from Differential Equations, given $\frac{dy}{dt} = g(t, y)$ and $y(t_0) = y_0$.

Euler's Method to approximate $y(t_f)$

Step 1. Divide $t_f - t_0$ into equal size sub-intervals, h

Step 2. Initialize $t_n = t_0$ and $y_n = y_0$.

Step 3. If t_n does not equal t_f then Go to Step 4

Step 4. $y_{n+1} = y_n + hg(t_n, y_n)$

Step 5 $t_n = t_n + h$, Go to Step 3.

$$\frac{dy}{dt} = .25y(10 - y), \quad y(0) = 2$$

Use Euler's Method to approximate $y(5)$. Try $h=0.5$ and $h=1$. Compare the results visually from $y(0) = 2$ to $y(5)$.

Exercises:

Consider the model $a(n+1) = ra(n)(1 - a(n))$. Let $a(0) = 0.2$. Determine the numerical and graphical solution for the following values of r . Find the pattern in the solution. Find Chaos.

1. $r = 2$
2. $r = 3$
3. $r = 3.6$
4. $r = 3.7$

Systems of Dynamical Systems

Let's consider systems of difference equations (DDS). For selected initial conditions, we build numerical solutions to get a sense of long term behavior of the system. For the systems that we will study, we will find their equilibrium values. We then explore starting values near the equilibrium values to see if the system

- a. remains close,
- b. approach the equilibrium value, or
- c. not remain close

What happens near these values gives great insight concerning the long-term behavior of the system. We can study the resulting pattern of the numerical solutions.

Competitive Hunter Problem

Competitive hunter models involve species vying for the same resources (such as food or living space) within their habitat. The effect of the presence of a second species diminishes the growth rate of the first species. We now consider a specific example concerning trout and bass in a small pond. Hugh Ketchum owns a small pond that he uses to stock fish and eventually plans to allow fishing. He has decided to stock both bass and trout. The fish and game warden tells Hugh that after inspecting his pond for environmental conditions he has a solid pond for growth of his fish. In isolation, bass grow at a rate of 20% and trout at a rate of 30%. The warden tells Hugh that the interactions for the food affect trout more than bass. They estimate the interaction affecting bass is $0.0010 \cdot \text{bass} \cdot \text{trout}$ and for trout is $0.0020 \cdot \text{bass} \cdot \text{trout}$. Assume no other changes in the habitat occur.

Model: Let $B(n)$ = the number of bass in the pond after period n , $T(n)$ = the number of trout in the pond after period n , and $B(n)T(n)$ = interaction of the two species.

$$B(n+1) = 1.20 B(n) - 0.0010 B(n) T(n)$$

$$T(n+1) = 1.30 T(n) - 0.0020 B(n) T(n)$$

Hugh initially buys 151 bass and 199 trout for his pond.

Resources

Much of my material comes from teaching courses at many different levels from high school to graduate level utilizing Elementary Mathematical Modeling: A Dynamical Approach by Jim Sandefur (published by Cengage-Brooks/Cole).

For other projects, contact the author.