SECOND SEMESTER BIOCALCULUS COMPUTER LABORATORY PROJECTS USING EXCEL AND MAPLE

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Introduction

We present a project that we use in the laboratory component of our second semester biocalculus course at Benedictine University. Computer technology is used throughout both semesters of biocalculus. We use a combination of Maple, Excel, and Berkeley Madonna in first semester. In the second semester, we add MATLAB. The project we present here is an investigation of models of logistic growth with harvesting using Maple. The project presented here is one of eight to ten projects assigned during the second semester of biocalculus. Other sample activities from both semesters are available on the author's web site:

http://www.ben.edu/faculty/tcomar/

The Logistic Model: A First Semester Project

Toward the end of the first semester, the students complete a project that addresses the *logistic population model*,

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \tag{1}$$

where r is a growth parameter, and K is the carrying capacity of the population, which is the maximal population size that can be sustained in the environment. The focus of the discussion is on understanding the interpretation of this differential equation and how to analyze it graphically. At this stage, students Maple to obtain plots of solution curves, direction fields, phase planes, and plots of solutions obtained by Euler's method. Through the use of phase planes and phase line analysis, students develop a conceptual and geometric understanding equilibria and basic stability analysis.

The Logistic Model with Harvesting

In the second semester of the biocalculus course sequence, we revisit the logistic model and consider what happens when the population undergoes harvesting. The modified model takes the general form

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \text{harvesting rate.} \tag{2}$$

The project investigates several different basic functions for the harvesting rate. We use H to denote the harvesting parameter. The units of H depend on the particular rate function. The studied harvesting rate functions are H (constant rate of harvesting), HN (constant effort harvesting), and $\frac{HN}{N+5}$ (a type II functional response). Biological questions that the students investigate are the following: (1) What is the biological interpretation of each of these three harvesting rate models? (2) Are the models realistic? If so, under what conditions are they realistic? If not, explain why not. (3) What model is the most realistic? Two other questions that involve both the biology and the mathematics are the following: (4) At what levels of harvesting will the population be sustained? (5) How does the dynamics of the system change when the harvesting parameter H varies? The first of these two questions can be used using stability analysis of equilibria. Graphical techniques for this analysis were introduced through the first semester project. In this second semester project, students learn an analytical technique for the stability analysis. The second question reinforces the concept of a parameter introduced earlier in the first semester biocalculus course. Here, the students will be able to observe that different values of H lead to different dynamical behavior. Through this second question, students are introduced to elementary bifurcation analysis.

Investigating The Model with Maple

We now illustrate how we can use Maple to help analyze the model and create useful figures. We focus our attention on the constant harvesting model, which is given by

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - H. \tag{3}$$

Step 1. Open a new Maple worksheet and load the *DETools* and *plots* packages with the commands:

with(DETools):with(plots):

Step 2. The equilibria are the solutions for N to the equation

$$\frac{dN}{dt} = rN\left(1-\tfrac{N}{K}\right) - H \ = 0 \tag{4}$$

To do this in Maple, type the following command:

$solve(r*N*(1-N/K)-H=0,N);$

Maple outputs the two equilibria:

$$\frac{1}{2}\frac{rK+\sqrt{r^2K^2-4rHK}}{r}, \frac{1}{2}\frac{rK-\sqrt{r^2K^2-4rHK}}{r}$$

From the above expressions for the equilibria, we can determine that there are two equilibria if $H < rK/4$, is one equilibrium if $H = rK/4$, and are no equilibria if

$H > rK/4$. Set $r = 0.1$ and $K = 100$ for the remainder of this activity. In Maple, enter:

$$r := 0.1 : K := 100 :$$

Step 3. Let $f(N) = rN(1 - N/K) - H$, and let N^* be one of the equilibrium values of N. The stability criterion states that if $f'(N^*) < 0$, then N^* is locally stable and if $f'(N^*) > 0$, then N^* is unstable. The criterion does not apply if $f'(N^*) = 0$. To apply the stability criterion, we first compute $f'(N)$:

$$\frac{d}{dN}\left(rN\left(1 - \frac{N}{K}\right) - H\right);$$

$$0.1 - 0.002N$$

We now evaluate $f'(N) = 0.1 - 0.002N$ at the equilibrium $N^* = (rK + \sqrt{r^2K^2 - 4rHK})/(2r)$:

$$subs(N = \frac{1}{2}\frac{rK + \sqrt{r^2K^2 - 4rHK}}{r}, 0.1 - 0.002N);$$

$$-0.01\sqrt{100 - 40H}$$

The case $H > 2.5$ is the case in which no equilibrium exists. This quantity is zero in the case that $H = 2.5$, and hence the stability criterion does not apply. The quantity is negative if $H < 2.5$ in which case the equilibrium $N^* = (rK + \sqrt{r^2K^2 - 4rHK})/(2r)$. is locally stable. Note that the two expressions for the equilibria are equal when $H = 2.5$. Using similar analysis, the other equilibrium is seen to be unstable of $H < 2.5$. The value $H = 2.5$ is called a *bifurcation value* because the dynamical behavior of the system changes as H passes through 2.5.

Step 4. We now create a dynamic phase portrait so that we can perform phase line analysis of the differential equation (3) as the parameter H varies. In the Maple worksheet we proceeds as follows:

Click on the Components tab and then select the Plot component. Now select the Slider component. Right-click on the plot window and select Component Properties to verify that it is named "Plot0". Click on OK. Right-click on the Slider and check Component Properties to verify that the slider is named "Slider0". Update the Slider values as follows:

Value at Lowest Position	0
Value at Highest Position	600
Current Position	0
Spacing of Major Tick Marks	100
Spacing of Minor Tick Marks	50

Make sure check marks are placed in EACH of the Options. Click on the [Edit] button. Delete all of the content preceded by "#" characters. (These are components.) In between the remaining two lines of code type the following:

$H0$:=GetProperty('Slider0','value'); $r := 0.1; K := 100;$
SetProperty('Plot0','value',*plot*($r * N * (1 - N/K) - H0/100$,$N$=0..120, y=-3..3,*labels*=["N","dN/dt"]));

Click [OK] on the Edit window and on the Slider Properties window. You have now created a slider that varies from 0 to 600.

The value of this slider is stored in the variable $H0$. We are interested in varying the parameter H in Equation (3) from 0 to 6. The parameter H is equal to $H0/100$. When we vary the slider from 0 to 600, we are really varying H from 0 to 6 as desired. The slider code we typed will plot the various phase portrait curves in the $N\frac{dN}{dt}$-phase plane. By dragging the arrow on the Slider, we can observe different phase plots. Notice that two equilibria appear when $H0 < 250$ ($H < 2.5$). In this case, the smaller equilibrium value is unstable, and the larger equilibrium value is locally stable. When $H0 = 250$ ($H = 2.5$), the is one equilibrium that is semistable. No equilibrium exists when $H0 < 250$ ($H > 250$). Figure 1 illustrates the case where $H0 = 100$ ($H = 1$).

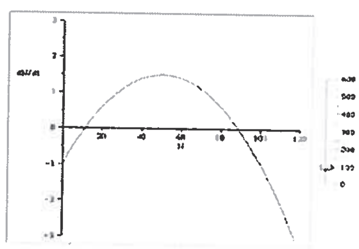

Figure 1: The phase portrait with the slider set to $H0 = 100$ ($H = 1$)

Step 5. We will now plot the bifurcation diagram for the differential equation with the parameter H. To make sure that H is treated as a variable by Maple, type:

$H :=' H' :$

With the parameter values of r and K fixed, enter the following command again:

$solve(r * N * (1 - N/K) - H = 0, N);$

Maple returns:

$$50 + 10\sqrt{25 - 10H}, 50 - 10\sqrt{25 - 10H}$$

Each of the solutions is a called a *branch curve*. In a bifurcation diagram, stable branches (corresponding to stable equilibria) are plotted as solid curves and unstable branches (corresponding to unstable equilibria) are plotted as dashed curves. Using the stability information you have previously found, plot these branches in the HN-plane, where H varies from -3 to 3 using solid or dashed curves as appropriate. Label all and classify bifurcation points.

The bifurcation diagram shown in Figure 2 can be created using the command:

display(*plot*(50.+10.*(25.-10.*H)^(1/2), H = -3 .. 3, *color* = *red*, *labels* = [“H”, “N”]), *plot*(50.-10.*(25.-10.*H)^(1/2), H = -3 .. 3, *color* = *blue*, *linestyle* = *dash*))

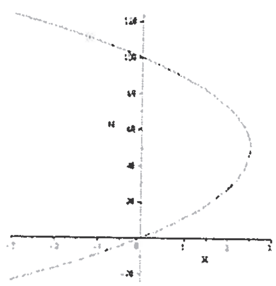

Figure 2: The bifurcation diagram with $r = 0.1$ and $K = 100$

The bifurcation point $H = 2.5$ is a saddle node bifurcation.

Step 6. To plot several solution curves for this differential equation with $H = 1$ and $r = 0.1$ and $K = 100$, we first define the differential in Maple using the command:

$$LogisticHarv := H \rightarrow diff(N(t), t) = 0.1 * N(t) * (1 - N(t)/100) - H;$$

We then use the following command to plot several solutions:

DEplot(LogisticHarv(1), $N(t), t = 0..40$, $[[N(0) = 10], [N(0) = 30],$ $[N(0) = 45], [N(0) = 60], [N(0) = 100]], N = 0..120);$

Note that this Maple command automatically provides the slope field as well.

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