ANIMATING CALCULUS CONCEPTS AND PROOF

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Today's students play and learn in the interactive animated visual worlds of the video iPod and iPhone. How can we, as teachers, bridge the gap between their dynamic worlds and the traditional calculus text? For many years now, we have devoted much of our research time to developing software tools that make calculus come alive for our students. We have added a set of new dynamic interactive visual tools to our software that help students to fully engage in learning key calculus concepts and proof.

Students and teachers can use the new Limit tool to visually explore limits, one-sided limits, continuity, and ε-δ proofs. Using the tool's step and animate buttons, one can move synchronized markers for both $(a-h,0)$ and $(a+h,0)$ on the x-axis, and for the corresponding points $(a-h, f(a-h))$ and $(a+h, f(a+h))$ on the function's graph to visually test if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and to determine if they are equal (see Figure 1). A table of values is also generated for a numerical verification of the limit. Using the tool's one-sided limit option and piecewise functions, it is easy to explore the concepts of a one-sided limit and continuity. See Figure 2.

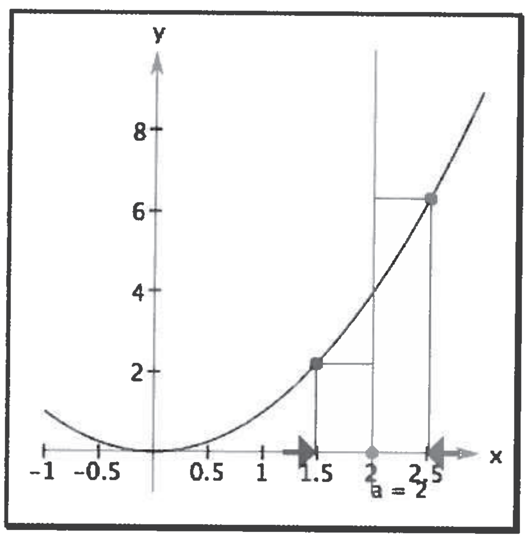


Figure 1 Two-Sided Limits

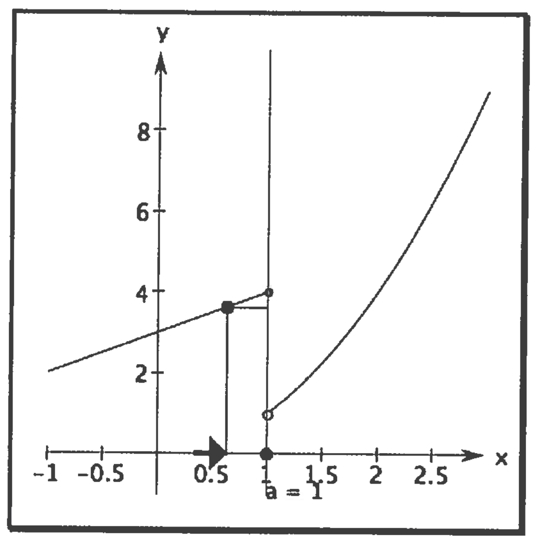


Figure 2 One-Sided Limits and Continuity

Similarly, the ε-δ Tool can be used to visually verify that for a given $\varepsilon > 0$, a value $\delta = \delta(\varepsilon)$ can be found such that $0 < |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$. Teachers can select an arbitrary ε and ask students to find a δ satisfying the definition of a limit. All of this is done visually by dragging the ε-δ regions. See Figure 3.

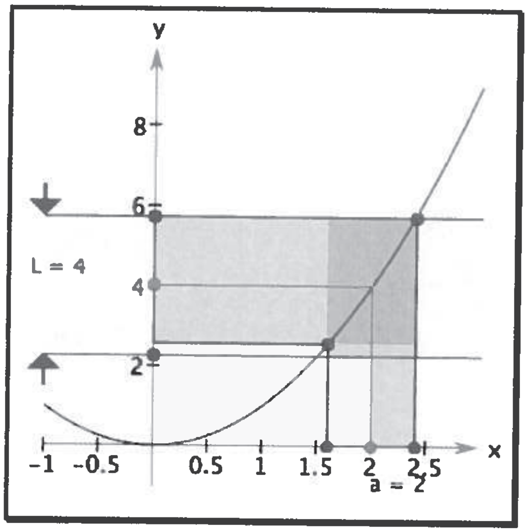


Figure 3 ε-δ Limit Proofs

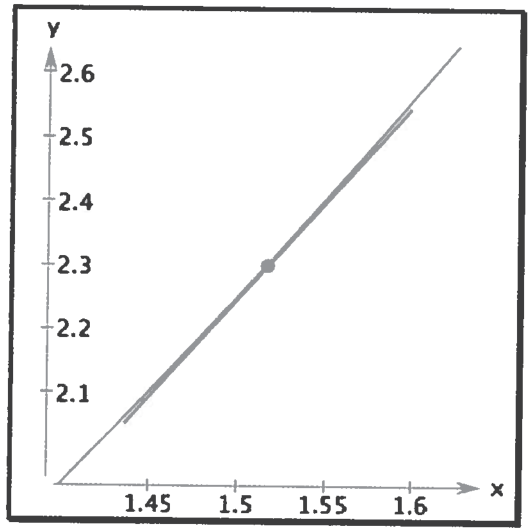


Figure 4 Local Linearity

The Zoom and Tangent tools can be used to introduce the concepts of linear approximation and local linearity. Using the Tangent tool, you can place a tangent line at any point along a curve. To investigate how well the tangent approximates the curve locally, use the Zoom tool's ability to zoom-in centered about the tangent. This visualization works for any function that is differentiable.

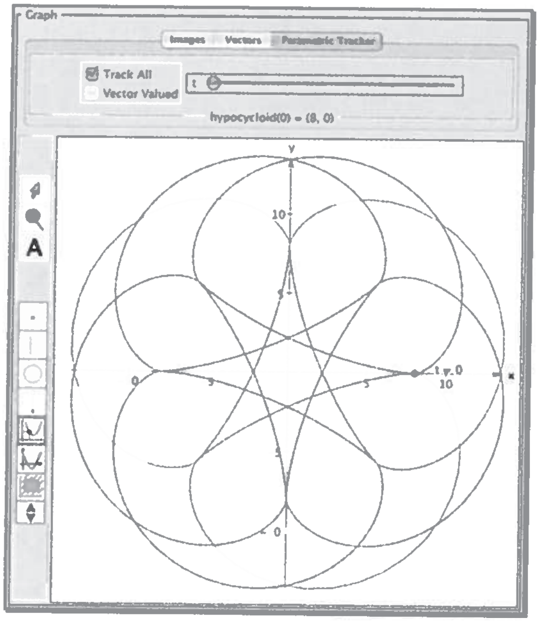
Conjectures galore can be made by visually exploring properties of hypocycloids and epicycloids. For example, teachers can ask their students, "Where and how often do the graphs of a hypocycloid and epicycloid meet"? To explore questions like these, enter the constants c1 (radius of fixed circle) and c2 (radius of rolling circle) along with the parametric equations for the hypocycloid

$$x(t) = (c_1 - c_2)\cos(t) + c_2\cos\left(\frac{(c_1 - c_2)t}{c_2}\right),\ y(t) = (c_1 - c_2)\sin(t) - c_2\sin\left(\frac{(c_1 - c_2)t}{c_2}\right)$$

and epicycloid

$$x(t) = (c_1 + c_2)\cos(t) - c_2\cos\left(\frac{(c_1 + c_2)t}{c_2}\right),\ y(t) = (c_1 + c_2)\sin(t) - c_2\sin\left(\frac{(c_1 + c_2)t}{c_2}\right)$$

into TEMATH. Plot the hypocycloid and epicycloids together and use the Parametric Tracker tool to simultaneously trace both curves and determine the points where they meet and the number of times they meet. Change the value of c1 to determine its effect on the curves. Make conjectures. See Figures 5 an 6 for example plots.



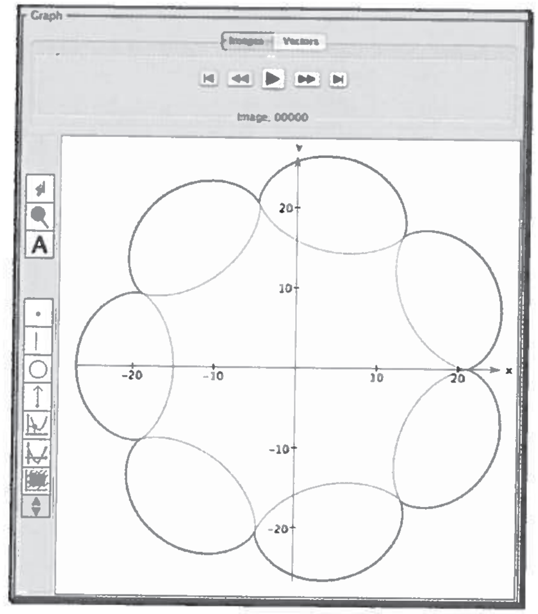


Figure 5 Hypocycloid and Epicycloid

Figure 6 Varying the Radius C1

TEMATH's Differential Equation Solver can be used to study the relationship between the solutions of a first-order linear system of differential equations and the eigenvalues and eigenvectors of the system. Consider the system

$$\frac{dx}{dt} = 3x + 2y \text{ and } \frac{dy}{dt} = 3x - 2y$$

This system has two real eigenvalues and the two corresponding eigenvectors, $\mathbf{V}_1 = (1, -3)$ and $\mathbf{V}_2 = (2, 1)$, can be used to produce two linearly independent straight-line solutions. Using TEMATH's interactive Differential Equation Solver, it is easy to draw a direction field and sample solution curves. Simply click the Direction Field button to draw the direction field and then click in the Graph panel to draw a solution curve using the clicked point as the initial condition. You can click a wide range of initial conditions to observe the properties of the solutions. The straight-line solutions generated by the

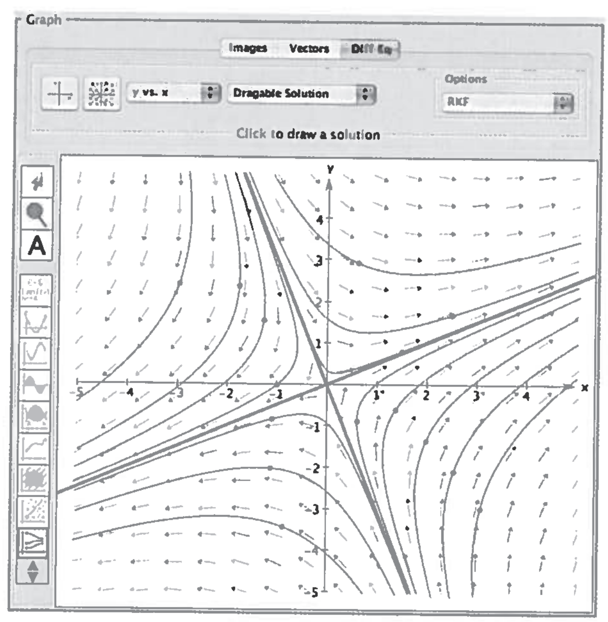


Figure 8 First-Order System of Differential Equations

eigenvectors can be entered into TEMATH as the functions $v_1(x) = -3x$ and $v_2(x) = x/2$. These can then be plotted on top of the direction field and solution curves. This produces a great image portraying the interrelationship between the solutions of the system and the straight-line solutions formed from the eigenvectors. See Figure 8.

A unique feature of our software is that you can import digital images into the Graph panel and mathematically model some physical phenomenon within the image. For example, on a recent trip to Cambodia, one of the authors found the shell of a Cambodian Giant Snail and exclaimed, "A perfect logarithmic spiral"! He brought the shell home, took a digital image of it, imported it into our software, and used polar coordinates to find the perfect spiral to model nature's shell. Figure 9 shows the shell of the snail and Figure 10 shows the excellent fit to its natural spiraling growth. This will be one more motivational example for our students to learn polar coordinates.

Figure 9 Cambodian Snail Shell

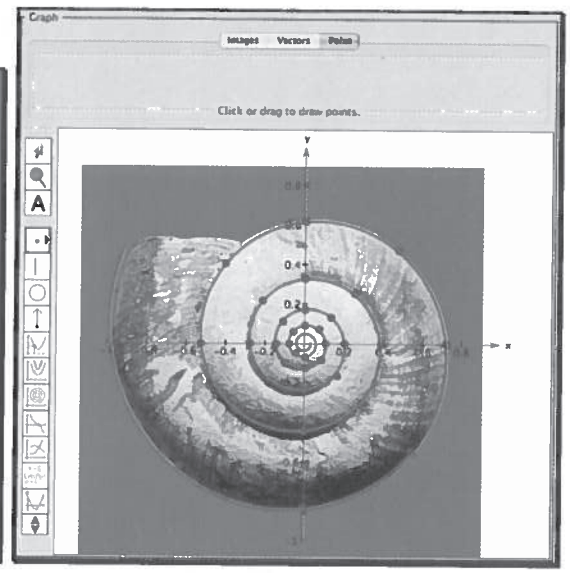


Figure 10 Logarithmic Spiral Fit

The Arc Length tool can be used to find the arc length of a function on an interval or to show a visual animation of the sum of the lengths of line segments converging to the arc length of the curve. The visual demonstration of an increasing number of line segments along a plotted curve will convince your students of the convergence of the sum of the lengths of the line segments to the length of the curve. You can increase the number of line segments by clicking the Step button or by simply clicking the Animate button and watching the convergence "show." Figure 11 shows a snapshot of the convergence process.

Finally, we've added a Level Curve tool that finds and plots a set of level curves of $f(x, y)$. Additionally, you can click at any point within the Graph panel to draw a level curve passing through that point or you can drag the level curve to observe how the shape changes as a function of z. There are also options for drawing a gradient at a point along a level curve or for drawing a color density plot. For example, a set of level curves for $f(x,y) = -4x/(x^2 + y^2 + 1)$ are shown in Figure 12.

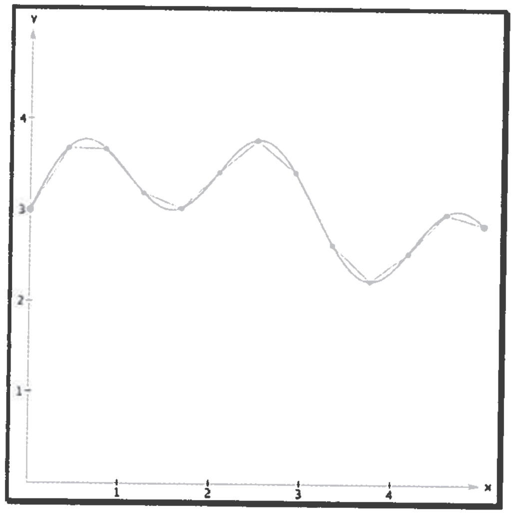


Figure 11 Arc Length Convergence

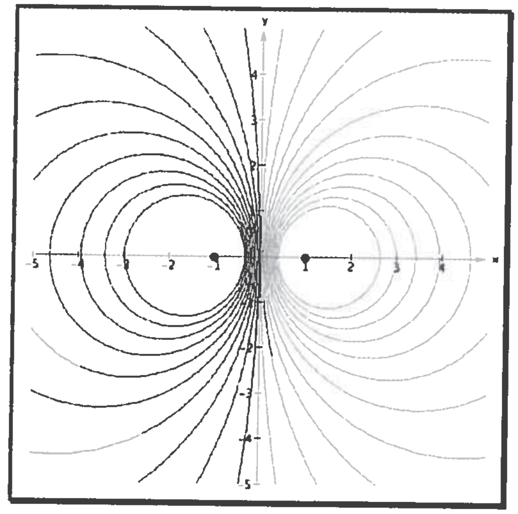


Figure 12 Level Curves

Our software is still in development. We plan to continue adding more visual tools for exploring and investigating other topics in calculus. Our goal is to have a common, uniform interface in which all these animations, mathematical experiments, explorations in applied problem solving, and visualizations can take place. Our interface design attempts to have the tools and information necessary for accomplishing a task readily available on the screen without searching for hidden functionality. We have an alpha version of our new cross-platform version of TEMATH available for testing. It runs on computers with Macintosh OS X, Linux, or current versions of Microsoft Windows/NT/XP/Vista operating systems that have a Java JVM 1.6 (or later installed). If you would like a copy of this new alpha version of TEMATH (V3.0a1), send us an email at rkowalczyk@umassd.edu or ahausknecht@umassd.edu and we'll send you a copy. We would gratefully appreciate any feedback on this version. Please report to us any bugs in the software and any suggestions for improving the interface or for adding new features. If you decide to use our software with your students, we would appreciate you sharing your experience with us. Once this version of TEMATH is more stable and ready for release, we will post it on our website **http://www2.umassd.edu/temath**.

Bibliography

[1] Adam Hausknecht and Robert Kowalczyk, "*Seeing is Believing – Visualizing Calculus*", Proceedings of the 20th Annual International Conference on Technology in Collegiate Mathematics, 2009.

[2] Adam Hausknecht and Robert Kowalczyk, "*Exploring Calculus Using Innovative Technology*", Proceedings of the 19th Annual International Conference on Technology in Collegiate Mathematics, 2008.