SOLVING POLYNOMIAL EQUATIONS VIA *EXCEL*

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I. Polynomial equations and their solutions
We often need to solve polynomial equations in our research or in courses in general mathematics. The linear and quadratic equations are easy. There are formulas for the cubic and quartic equations, though less familiar. There are no algebraic methods to solve the quintic or higher order equations. There are a host of numerical solutions to solve these equations and other nonlinear equations as well. The disadvantage is that we need to relearn these methods and we need to program these algorithms in our computers before they can be implemented.

In contrast, the advantages of *Excel* are that it is readily available in every personal computer, and no programming is required to solve the equations. Excel has the built-in features *Goal Seek* and *Solver* that can do a lot of mathematical operations. Here we will provide some examples on the use of Excel to solve typical polynomial equations up to order 5 or higher. It will then become apparent that the utility is very user-friendly and with results obtainable to any desired accuracy.

For comparison, we will mention some examples done by classical methods, like linear interpolation and Newton's method. The order of precision is about the same, but the ease in the use of Excel over the other methods is no comparison.

II. Examples
A. In Excel, we tabulate corresponding values of x and y, and plot the graph of the

function. The x-intercepts give us the real roots of the equation. We know from the intermediate-value theorem that a polynomial has a root where the y-values change from a negative to a positive value, or from a positive to a negative value; in particular, an odd-order polynomial always has at least one real root.

(1) *Cubic equation:* $y = 3x^3 - 7x^2 + 3x + 1$ [adapted from Bourg, 2006].
In this example, x = 1 is an obvious root; the other roots are found from the graph, or from the table of values (Figure 1).

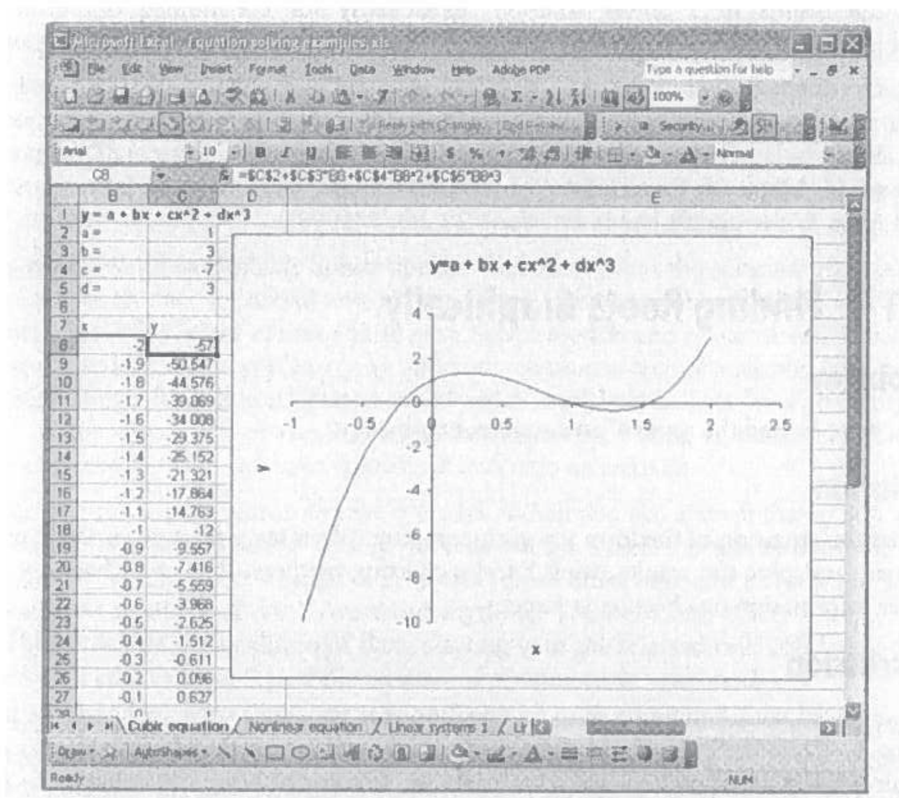

Figure 1. Cubic Polynomial

At this point, we have several options: to interpolate between the positive and negative tabulated values, or do Newton's approximation. These are what we mention in the classical methods. If we are not content with the residual value of y from zero, we can change the options in *Goal Seek* (Figure 2). The default values for *Goal Seek* are 0.001 for the residual value (from your target value of 0) and 100 maximum iterations (Figure 3). For example, using the default values, we got x = -0.21525, with residual -1.60131E-05. When we changed the residual value to 1.0E-6 at 5000 maximum iterations, the other root found was x = -1.548584, with residual -1.26121E-13. The result was practically instantaneous.

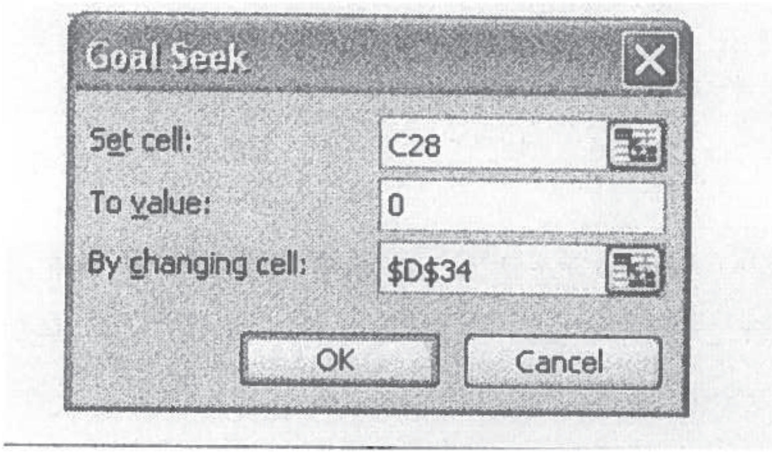

Figure 2: *Goal Seek* parameters

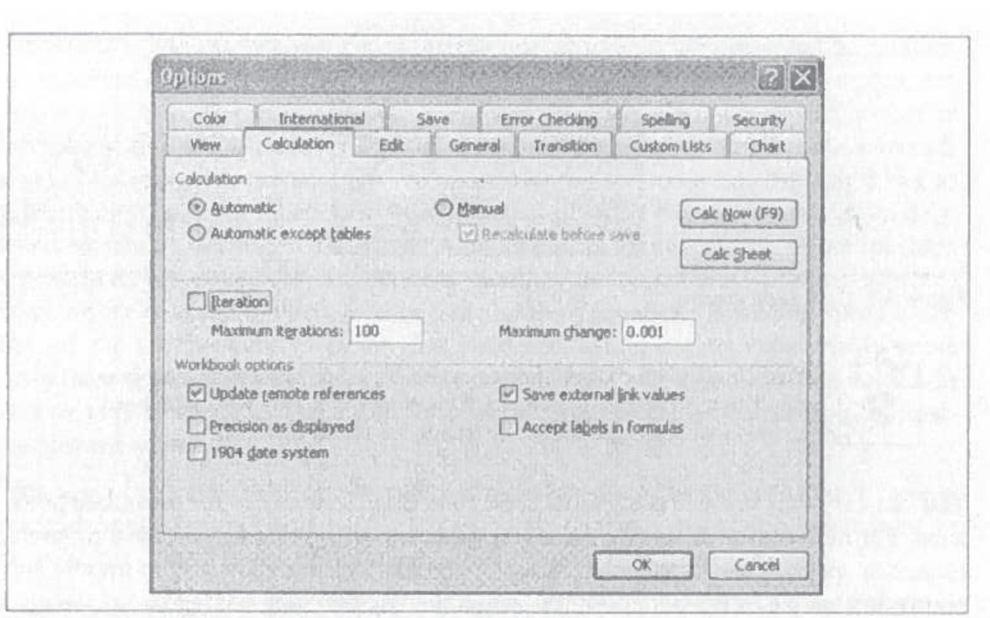

Figure 3: *Goal Seek* options

The solution can also be obtained using *Solver* (Figure 4). There are two parameters to assign: "Equal To" to 0 (the righthand-side of the equation), and "By Changing Cell" to the x-value you pick nearest the zero value of *y*. The results are comparable to the *Goal Seek* results.

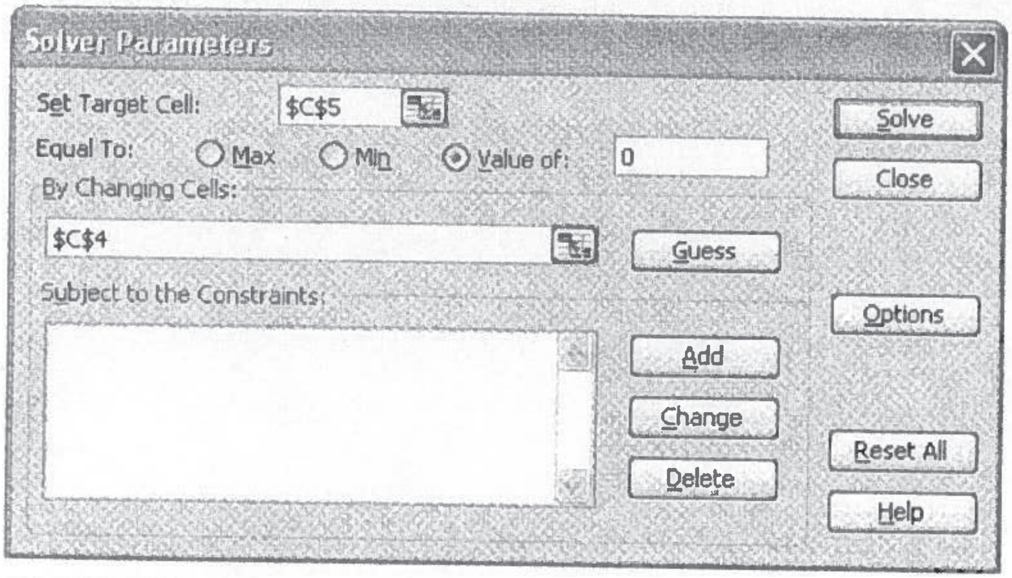

Figure 4: *Solver* parameters

(2) The next example is a quintic equation: $y = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16$ [from Sullivan, 2008, p381]. From the tabulated values and the graph that Excel provides, $x = 1$ is a root, and $x = 2$ is a double root. Two roots must be complex, but these are not given by Excel offhand. This suggests a topic for a future study: How to obtain the complex roots of an equation?

(3) Another quintic polynomial: $y = x^5 - x^3 - 1$ [Sullivan, p384].
This quintic has only one real root: at x = 1.236506, with residual 8.28668E-10.

(4) A sextic equation: $y = 3x^6 - 4x^4 + 3x^3 + 2x^2 - x - 3$ [Sullivan, p378].
This equation has two real roots: at $x = 1$, and at x = -1.36557, with residual 6.84E-09.

(5) Our final example is the celebrated monster equation of order $n = 45$ (Figure 5).
This has an interesting twist in the history of mathematics, as recounted by Pesic (p 45). In 1593, the Belgian mathematician Adriaan van Roomen challenged 'all the mathematicians of the whole world' to solve this monster equation of the 45th degree. The Dutch ambassador, paying a call on Henry IV, ironically offered the king his condolences that there were no French mathematicians up to the challenge. Stung, Henry called on Francois Viete, who sat down with the problem and was able within a few minutes to find the positive roots of the equation. Viete happened to be working on the very same equation from his studies in trigonometry; the slightest change in the coefficients would have proved insurmountable.

Box 2.7
Adriaan van Roomen's test problem

Solve

$$
\begin{aligned}
&x^{45} - 45x^{43} + 945x^{41} - 12{,}300x^{39} + 111{,}150x^{37} - 740{,}459x^{35} \\
&\quad + 3{,}764{,}565x^{33} - 14{,}945{,}040x^{31} + 469{,}557{,}800x^{29} \\
&\quad - 117{,}679{,}100x^{27} + 236{,}030{,}652x^{25} - 378{,}658{,}800x^{23} \\
&\quad + 483{,}841{,}800x^{21} - 488{,}494{,}125x^{19} + 384{,}942{,}375x^{17} \\
&\quad - 232{,}676{,}280x^{15} + 105{,}306{,}075x^{13} - 34{,}512{,}074x^{11} \\
&\quad + 7{,}811{,}375x^{9} - 1{,}138{,}500x^{7} + 95{,}634x^{5} - 3{,}795x^{3} \\
&\quad + 45x = K,
\end{aligned}
$$

where K is a given number.

Figure 5: The 'monster' equation

Using Excel, with the righthand-side $K = 0$, the positive roots are immediately found: $x = 0.139513, 0.415907$, and 0.512176, with respective residuals of -9.1E-09, -6.00E-13, and -1.05E=11. It takes a great deal more time entering the equation than getting the results.

B. Classical methods

The classical methods we mention here are both iterative methods. Around the point where the y-value changes sign lies the root of the equation. Newton's method is based on the idea of estimating the actual intersection of the curve by the intersection of the tangent to the curve at some initial value. Subsequent estimates are made repeatedly until the estimate converges on the true value. The method requires an initial value estimate and the derivative of the function at the estimate:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Another iterative method is the linear interpolation or secant method. It has the advantage of not requiring the derivative of the function, but we do need two initial guesses of the root in order to construct a secant line instead of the tangent line for making x estimates:

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}.$$

Suffice to note that the order of accuracy can be set to our requirements and can be made as efficient as the order of residuals for the solutions obtained previously by *Goal Seek* and *Solver*.

III. Suggested future work

The work illustrated here suggests further work that we may carry out in *Excel,* through its tools *Goal Seek* and *Solver.* One is a direct continuation of the problem of finding the roots of a polynomial equation, not only the real roots, but including the complex roots as well. For sure the graphing procedure will not be much help so we will have to rely more on the analytic solution.

Another very important and more practical problem is the solution of nonlinear equations, which also occurs rather often in research and in our mathematics courses. We plan to continue the solution of these problems in the near future.

IV. Conclusion

We have demonstrated that polynomial equations can be readily solved by *Excel* using its tools *Goal Seek* and *Solver.* The process is very user-friendly because Excel easily tabulates the function and makes a graph of the function. The order of precision can be set to any desired accuracy, and on this account is just like the other classical methods we used to employ for these problems. But the ease in the use of Excel is a definite advantage because it requires no programming. And the fact that this utility is readily available is its most attractive feature.

References:

1. Bourg, DM, 2006. *Excel Scientific and Engineering Cookbook.* Sebastopol, CA: O'Reilly Media, Inc.
2. Pesic, P, 2003. *Abel's Proof.* MA: MIT Press.
3. Sullivan, M, 2008. *Algebra and Trigonometry.* NJ: Prentice-Hall.