Generating Undergraduate Research Problems in Difference Equations with Computer Algebra

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1 Introduction

At the University of Rhode Island, we offer a senior level undergraduate course in difference equations, which runs every Spring semester, in addition to graduate level courses and seminars. Our undergraduate course is attended by an average of 30 students that come from natural sciences, mathematics, engineering and even social sciences. A key activity of the course is a research project that the students must carry out and present to the class at the end of the semester. One of the challenges the instructor of difference equations faces is that of the generating enough problems with interesting characteristics for a relatively large class.

Here we are concerned with invariants for difference equations. The theory is based on the existence of an expression, called an invariant or first integral, that remains constant along solutions of a difference equation and which reveals much about the behavior of solutions of the equation.

We discuss here the QRT Algorithm for generating large families of difference equations that have invariants. We begin in Section 1 with examples of difference equations and invariants. Then in Section 2 we present the QRT Algorithm with an example.

First Difference Equation with invariant. Theon of Smyrna (ca. 100) studied what appears to be the first difference equation with an invariant. It is at the heart of his method for approximating $\sqrt{2}$ with rational numbers. Theon of Smyrna's approach is as follows. Choose $x_1 > 0$, $y_1 > 0$, and set

$$
\begin{array}{rclcrcl}
x_2 & = & x_1 + y_1 & \quad & y_2 & = & 2\,x_1 + y_1, \\
x_3 & = & x_2 + y_2 & \quad & y_2 & = & 2\,x_2 + y_2, \\
& \vdots & & & & \vdots & \\
\end{array}
$$

$$x_{n+1} = x_n + y_n \qquad y_{n+1} = 2\,x_n + y_n$$

For example, $x_1 = 1$ and $y_1 = 1$ give $(1,2)$, $(2,3)$, $(5,7)$, $(12,17)$, ... At first glance, the numbers look unremarkable. However, one can easily verify that $y_n^2 - 2\,x_n^2 = (-1)^n$ for $n = 1, 2, 3, \ldots$. Let us define the function (called *invariant*)

$$I(x,y) := \left(y^2 - 2\,x^2 - 1\right)\left(y^2 - 2\,x^2 + 1\right)$$

Then $I(x_n, y_n) = 0$ and

$$y_n^2 = 2\,x_n^2 \pm 1\,, \quad \text{i.e. ,} \quad \left(\frac{y_n}{x_n}\right)^2 = 2 \pm \left(\frac{1}{x_n}\right)^2$$

Since $x_{n+1} > 2\,x_n$ (so in particular $x_n \to \infty$), we have that knowledge of the invariant allows us to conclude $y_n/x_n \to \sqrt{2}$.

Using an invariant to approximate $\sqrt{N}$. For $x_0 > 0$ and $y_0 > 0$, consider the "arithmetic-harmonic mean" system of nonlinear difference equations:

$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \frac{2}{\dfrac{1}{x_n} + \dfrac{1}{y_n}}, \qquad n = 0, 1, 2, \ldots \tag{1}$$

It can be shown that $\{x_n\}$ and $\{y_n\}$ are monotone and bounded, thus they converge, say $\lim x_n = a$ and $\lim x_n = b$. Furthermore, upon taking limit in either equation of (1) we obtain $a = \frac{a+b}{2}$, from which we conclude $a = b$. But the actual values of a and b cannot be derived directly from (1). Let us consider the product $x_{n+1}\,y_{n+1}$:

$$x_{n+1}\,y_{n+1} \;=\; \frac{x_n + y_n}{2}\,\frac{2}{\frac{1}{x_n} + \frac{1}{y_n}} \;=\; \frac{x_n + y_n}{2}\,\frac{2\,x_n\,y_n}{x_n + y_n} \;=\; x_n\,y_n \tag{2}$$

which shows that the product $x_n\,y_n$ remains constant as n varies in $\mathbb{N}$. If we define $I(x,y) = x\,y$, from equation (2) we have $I(x_{n+1}, y_{n+1}) = I(x_n, y_n), \quad n = 0, 1, 2, \ldots$. In particular, $I(x_n, y_n) = I(x_0, y_0)$, that is, $x_n\,y_n = x_0\,y_0$. Combine with $\lim x_n = a$ and $\lim y_n = a$ to obtain $a^2 = x_0 y_0$. Thus the limit a of $\{x_n\}$ and of $\{y_n\}$ equals the product of x_0 and y_0. For example, set $x_0 = 1$ and $y_0 = 2$, so that $N = a^2 = 2 \cdot 1 = 2$. Then, both $x_n,\ y_n \to \sqrt{2} \approx 1.41421356237309504880\ldots$ where the convergence is order two.

n	x_n	y_n	Error
0	1.	2.	6×10^{-1}
1	1.5	1.33333333333333333333	8×10^{-2}
2	1.41666666666666666666	1.41176470588235294117	2×10^{-3}
3	1.41421568627450980392	1.41421143847487001733	2×10^{-6}
4	1.41421356237468991062	1.41421356237150018697	2×10^{-12}
5	1.41421356237309504880	1.41421356237309504880	$\epsilon_{machine}$

An example where the invariant is the quantity to be approximated. For $a_0 > 0$ and $b_0 > 0$, consider

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n\,b_n}, \qquad n = 0, 1, 2, \ldots \tag{3}$$

System (3) is called the *"aritmetic-geometric mean iteration of Gauss and Legendre"* One can show that a_n's and b_n's are monotone and bounded, and that they converge to the same limit. The common limit is known as the *arithmetic-geometric mean*, usually represented with the symbol $AGM(a_0, b_0)$. By using some transformations one can show that the *complete elliptic integral of the first kind*

$$I(a,b) = \int_0^{\pi/2} \frac{dt}{\sqrt{a^2 cos^2 t + b^2 sin^2 t}}, \tag{4}$$

satisfies $I(a_{n+1}, b_{n+1}) = I(a_n, b_n) = \cdots = I(a_0, b_0)$, which means that $I(a,b)$ is an *invariant* of Equation (4). Gauss' formula is also valid. It is: $I(a_0, b_0) = \frac{\pi}{2} AG(a_0, b_0)$. That is, one may use a_n and b_n to approximate the elliptic integral $I(a_0; b_0)$.

Definition of Invariant (Autonomous Case). *Let* $\mathbf{f} : D \to D$ *be a continuous function, where* $D \subset \mathbb{R}^k$. *A nonconstant continuous function* $I : D \to \mathbb{R}$ *is an* invariant *for the difference equation* $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$, $n = 0, 1, \ldots,$ $\mathbf{x}_0 \in D$ *if* $I(f(\mathbf{x})) = I(\mathbf{x})$, *for every* $\mathbf{x} \in D$.

Example: Lyness' Equation. For $A > 0$ consider

$$x_{n+1} = \frac{A + x_n}{x_{n-1}}, \quad n = 1, 2, \ldots; \quad x_0 > 0, \ x_1 > 0 \tag{5}$$

The equilibrium point is obtained as follows:

$$x_{n+1} = \frac{A + x_n}{x_{n-1}} \quad \Rightarrow \quad \overline{x} = \frac{A + \overline{x}}{\overline{x}} \quad \Rightarrow \quad \overline{x} = \frac{1 + \sqrt{1 + 4A}}{2}$$

The standard way of studying stability of the equilibrium is to check whether the roots of the characteristic equation of a linearization of Eq. (5) about the equilibrium are inside the unit disk. In this case the characteristic equation can be shown to be $\lambda^2 - \frac{1}{\overline{x}} \lambda + 1 = 0$. An invariant for equation (5) is

$$I(x, y) = \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)(A + x + y)$$

The above invariant may be obtained by hand calculations, or by invoking a special function of the software package *Dynamica* ([1], [2]), written in Mathematica language by the authors of this note. By visualizing the invariant I one may conjecture that it has an isolated minimum attained at the equilibrium point of Eqn. (5). In fact, one can prove this either by hand calculation or by using *Dynamica*, see [1]. This fact, in view of Morse's Lemma [6], implies that all solutions to Eqn. (5) belong to the family of closed simple curves, which shows that all solutions are bounded and that the equilibrium is stable, see [1].

In general, students may use an invariant to investigate and establish key properties of difference equations, such as

- boundedness of solutions (i.e., $x_n < M$).
- persistence of solutions (i.e., $0 < \delta < x_n$).
- stability of the fixed point $\overline{x}$.
- the limiting value of solutions $\{x_n\}$ when they converge.

2 Generating difference equations with invariants

The following theorem gives a way of generating many difference equations that have invariants (see [3], [4], and [5]).

The QRT Theorem *Let A and B be 3 by 3 real symmetric matrices, and let*

$$\mathbf{x} := \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \quad \mathbf{y} := \begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix}, \quad \begin{pmatrix} f(x) \\ g(x) \\ h(x) \end{pmatrix} := (A\,\mathbf{x}) \times (B\,\mathbf{x})$$

Then, for $x_0 > 0$, $x_1 > 0$, the difference equation

$$x_{n+1} = \frac{f(x_n) - x_{n-1}\,g(x_n)}{g(x_n) - x_{n-1}\,h(x_n)} \quad \text{has the invariant} \quad I(x,y) = \frac{\mathbf{y}^t\,A\,\mathbf{x}}{\mathbf{y}^t\,B\,\mathbf{x}} = \frac{\sum A_{ij} y^i x^j}{\sum B_{ij} y^i x^j}$$

An interesting fact about this result is that the resulting equation and the corresponding invariant depend on twelve parameters, namely the entries of the input matrices A and B. A particular choice of A and B gives Lyness' equation, for example. The formula given in the QRT Theorem is easily implemented in any computer algebra system. Thus a large collection of examples may be generated without effort for teaching purposes. The following is an implementation in *Mathematica*.

INPUT: A, B (real symmetric 3 by 3 matrices)

```
QRTSymm[A_, B_] :=
  Module[{f1, f2, f3, qrteqn, invariant},
   {f1[x_], f2[x_], f3[x_]} = Cross[A.{x^2, x, 1}, B.{x^2, x, 1}];
   qrteqn =
    x[n + 1] == Simplify[(f1[x[n]] - x[n - 1] f2[x[n]]) /
       (f2[x[n]] - x[n - 1] f3[x[n]])];
   invariant =
    Simplify[(A.{y^2, y, 1}).{x^2, x, 1} / (B.{y^2, y, 1}).{x^2, x, 1}];
   Return[{QRTeqn → qrteqn, QRTInvariant → invariant}];
  ];
```

Here is an actual output of a Mathematica session, where a difference equation and corresponding invariant are generated from symbolic input.

```
In[2]:= ans = QRTSymm[ (a b c)   (0 0 0)
                       (b 0 e) , (0 1 0) ]
                       (c e f)   (0 0 0)

Out[2]= {QRTeqn → x[1 + n] == (f + e x[n] + c x[n]^2) / (x[-1 + n] (c + b x[n] + a x[n]^2)),
         QRTInvariant → 1/(x y) (f + e y + c y^2 + x (e + b y^2) + x^2 (c + y (b + a y)))}
```

Conclusion. Invariants are a powerful tool for analyzing either local or global behavior of solutions to difference equations, and in many cases they completely reveal the global character of solutions of these equations, such as convergence to the equilibrium, boundedness, stability, existence of periodic solutions, etc. In this paper we presented a method of creating a large family of rational difference equations together withe their invariants, which can be used in a classroom setting for teaching or for research purposes as well.

References

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