DEMONSTRATING CALCULUS CONCEPTS USING THE GEOMETER'S SKETCHPAD®

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Introduction

For the better part of the last decade, we have been using computer algebra systems to demonstrate graphical concepts in our Calculus classes. The animations we create (mostly in Maple) are helpful to the students—but we sometimes find ourselves wanting to interact directly with the graphics, and so we have recently started using The Geometer's Sketchpad® for some of these dynamic demonstrations. For example, students can *see* us dragging one point closer to another, or changing the size of an angle. We can drag points quickly or slowly, depending on how students respond to our questions. We can ask students to form conjectures, and then quickly adjust a sketch to test those conjectures.

Some of the sketches we have created include demonstrations of
- the relationship between ε and δ in the definition of limit
- the relationship between the unit circle and the graph of $\sin\theta$
- $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta}$
- the relationship between secant lines and tangent lines
- the relationship between graphs and differentiability
- the relationship between the graph of $f(x)$ and the graph of $f'(x)$
- the relationship between the graph of $f(x)$ and the graph of $f(x+h)+k$

In the remainder of this paper, we will describe how to use Sketchpad to create several of these sketches. We assume that the reader is familiar with basic constructions in Sketchpad.

The ε-δ Definition of Limit

- Open a new sketch, and use the Graph menu to show the (square) grid. Drag the origin to recenter the graph, and drag the point $(1,0)$ to resize the graph.

- Use the Graph menu to plot a new function. In the example below, we used $f(x) = \frac{1}{4}x^2$.
- From the Graph menu, construct new parameters $zero = 0$ and $a = 1.5$.
- From the Measure menu, calculate $f(a)$. (In the Calculate dialog box, click on the expression for $f(x)$ and the expression for a.) Right click on the equation $f(a) = 4.00$, choose "Properties," and change the label to L.
- Highlight the expressions for a and $zero$, and choose "plot as (x, y)" from the Graph menu. Plot $(0, L)$ similarly. Label the two points a and L, respectively, and hide the parameter $zero$.
- Construct points on the x- and y-axes labeled $a + delta$ and $L + epsilon$, respectively.
- Construct lines through $a + delta$ and $L + epsilon$ parallel to the y- and x-axes, respectively.
- Rotate $a + delta$ and the vertical line through it 180° about a. Rotate $L + epsilon$ and the horizontal line through it 180° about L. Label the resulting points $a - delta$ and $L - epsilon$.
- Highlight $L + epsilon$ and L, and measure the coordinate distance between them. Change the label on this value to $epsilon$. Measure and label the value of $delta$ similarly.
- Adjust colors, font sizes, and line thicknesses for optimal visibility in your classroom.
- Use this sketch in class by dragging the point $L + epsilon$ to a desired value of $epsilon$. Then drag the point $a + delta$ to a value of $delta$ that "works" for that ε. We ask the students to tell us when to stop moving $a + delta$.
- Change the definitions of $f(x)$ and a by double clicking on their values. Everything else will change accordingly.
- Caution: When you resize the graph, a and L will maintain their values, but $a + delta$ and $L + epsilon$ will not.
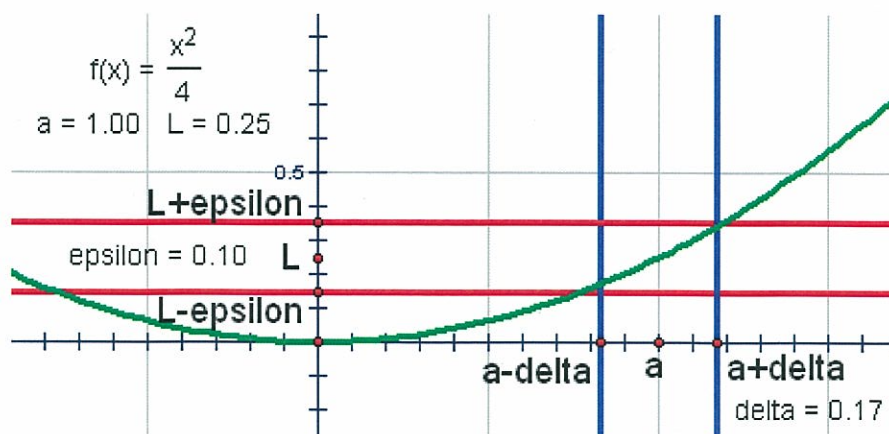

Figure 1: The $\varepsilon - \delta$ Definition of a Limit

The limit of $\dfrac{\sin\theta}{\theta}$ **, as** θ **approaches 0**

- Open a new sketch, and use the Graph menu to show the (square) grid. Choose "Preferences" from the Edit menu, and change the angle units to radians.
- Drag the origin to recenter the graph, and drag the point $(1,0)$ to resize the graph. Label the origin O, and label the point $(1,0)$ A.
- Construct the unit circle, by center and point.
- Construct a point on the circle. Label this point B.
- Construct the line segments from O to A, and from O to B.
- Measure angle AOB. To get a decimal value for this measurement, choose "Calculate" from the Measure menu, click on the angle measurement, and add 0.0. Then hide the original angle measurement. Right click on the decimal measurement and choose "properties" to change the label to *theta*.
- Calculate sin(*theta*). (In the Calculate dialog box, choose the sine function, and then click on the value of *theta*.)
- Construct the perpendicular line from B to the x-axis. Construct the perpendicular line segment from B to the x-axis, and hide the perpendicular line.
- From the Measure menu, calculate $\dfrac{\sin\theta}{\theta}$.
- Adjust colors, font sizes, and line thicknesses for optimal visibility in your classroom.
- Use this sketch in class by dragging the point B closer to the x-axis. Remind students that θ is the length of the arc from A to B as well as the measure of angle AOB, and that $\sin\theta$ is the vertical distance from the x-axis to B. As θ gets close to 0, students can see that those distances become almost equal. Students can also watch as the ratio $\dfrac{\sin\theta}{\theta}$ gets closer and closer to 1. Remember to move the point B below the x-axis, and slowly move it up, so that students see what happens when θ approaches 0 from the left as well as from the right.
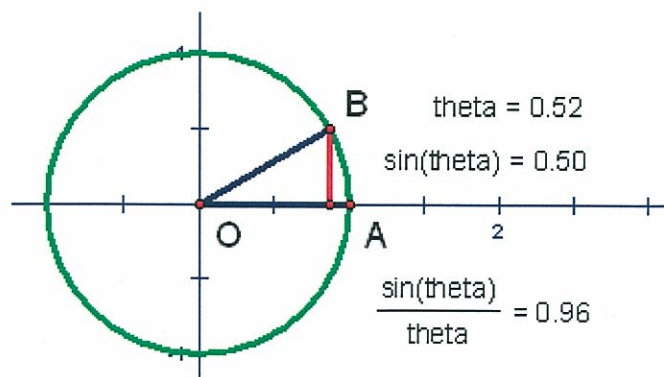

Figure 2: $\displaystyle\lim_{\theta\to 0}\dfrac{\sin(\theta)}{\theta}$

The Graph of $f'(x)$

- Open a new sketch, and use the Graph menu to show the (square) grid. Drag the origin to recenter the graph, and drag the point $(1,0)$ to resize the graph.
- Use the Graph menu to plot a new function. In the example shown below, we used $f(x) = \dfrac{x^3}{4} - \dfrac{x^2}{2} - x + 1$. (If you wish to provide additional examples using a single sketch, it is possible to create sliders for the coefficients of a cubic polynomial, and adjust the shape of the graph using the sliders – see the Help menu.)
- Construct a point on the function plot, somewhere on the left side of the graph, and label it O.
- Use the Measure menu to measure the abscissa and ordinate of O.
- Choose "Calculate" from the Measure menu, click on the abscissa measurement, x_O, and add .01. This defines a new x-value very close to the x-coordinate of point O.
- Now calculate the function's value at this new x-value. To do this, choose "Calculate" from the Measure menu, and then click on the formula for f, and then on the formula for your new x-value.
- Click on the new x-value and on the new function value, and then choose "Plot as (x, y)" from the Graph menu. This plots a new point on the graph of f that is very close to the point O.
- Construct the line through O and the newly plotted point. Then use the Display menu to hide the newly plotted point. The remaining line will then be a very good imitation of the tangent line to the graph of f at the point O.
- You may now hide all of the measurements except for x_O, the abscissa of point O.
- Select the "tangent" line and choose "Slope" from the Measure menu. Relabel this slope as "Slope of tangent line," by right-clicking it and choosing "Properties."
- Select x_O and the measurement of the slope of tangent line, and then plot the resulting point by choosing "Plot as (x, y)" from the Graph menu. (If you can't see the resulting point, it is probably outside the current window. Scroll a little bit up or down using the scroll bar, or resize the graph by dragging the point $(0,1)$.)
- Select only the newly plotted point (which is a point on the graph of f'), and choose "Trace Plotted Point" from the Display menu.
- Adjust colors and line thicknesses for optimal visibility in your classroom.
- Use this sketch in class by dragging the point O around on the graph of f, and watching as points on the graph of f' are traced. Pause at key locations and have students observe the relation between the slope of the tangent line and the y-coordinate of the corresponding point on the graph of f'.
- If you wish to run through the experience a second time, you should first move the point O to a convenient starting point, and then choose "Erase Traces" from the Display menu.

As an optional extension of this demonstration, you can animate the motion of the point O along the graph of f. You can stop the animation at any time while you discuss the relation between the slope of the tangent line and the corresponding point on the graph of f', and then resume the animation, just by clicking the animation control button. To add the animation, follow the additional steps below.

- Be sure that only point O is selected. From the Edit menu, choose "Action Buttons ... Animation." Select the "Animate" tab, and choose **forward** direction, **once only**, and **fast speed**.
- You may also select the "Label" tab, and change the label of the resulting button to something more descriptive, such as "Show tangents and plot derivative." (If you have already created the button, simply right-click on it and choose "Properties," and then make the desired changes.
- If you find that "fast speed" is still too slow, you may change the speed of the animation to "other." Try values like 5 or 8 for the speed.
- Caution: It is best not to let the animation run outside the visible portion of the plane. If you do, it may then be difficult to locate point O to drag it back to its original position.
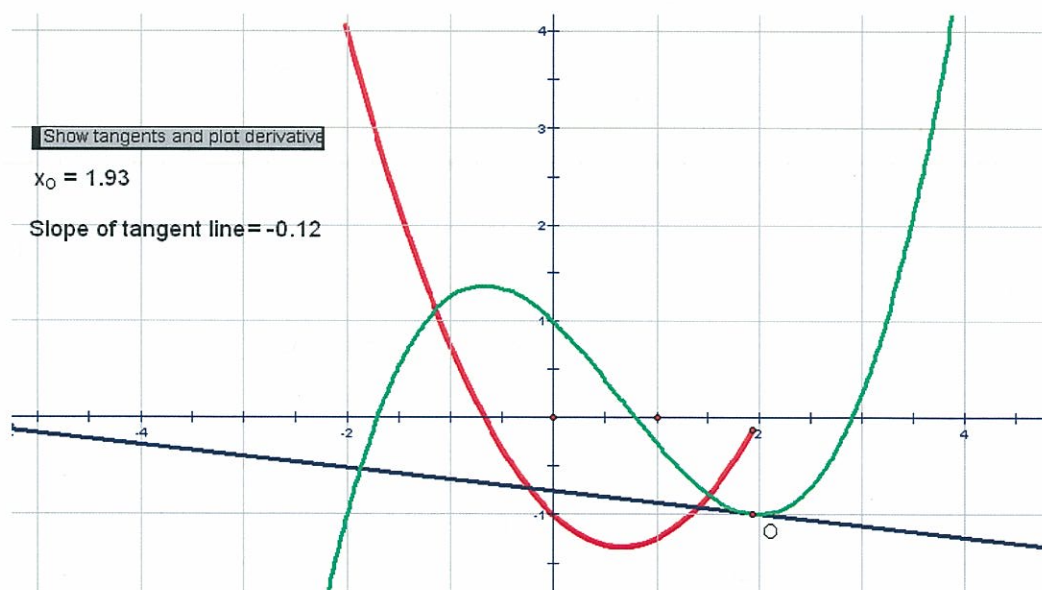

Figure 3: The Graph of the Derivative

Conclusion

We will continue to use many of our animations and other demonstrations from computer algebra systems, particularly for functions of two variables. But in cases where direct interaction with a graph will enhance students' understanding, Sketchpad has become our tool of choice.