EXPLORING DIFFERENTIAL EQUATIONS WITH MATHEMATICA

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1. Introduction

The *Mathematica* function DSolve finds analytical solutions of differential equations. This paper, which is based on a presentation at ICTCM, 2007, aims to give a comprehensive introduction to the methods implemented in DSolve and to show some interesting applications of this functionality. At present, DSolve can find exact solutions for the following types of equations:

1. Ordinary differential equations (ODEs), in which there is only one independent variable and one or more dependent variables. For example, the following system of ODEs has one independent variable s and three dependent variables x, y, and t. This system describes a plane curve through the origin whose curvature is equal to the parameter value s at every point. The solution is given in terms of Fresnel functions and the curve is a Cornu spiral (see Figure 1)::

InputForm[sol = DSolve[{$x'(s) = \cos(t(s))$, $y'(s) = \sin(t(s))$, $t'(s) = s$, $x(0) = 0$, $y(0) = 0$, $t(0) = 0$}, {x, y, t}, s]]

```
{{t -> Function[{s}, s^2/2], x -> Function[{s}, Sqrt[Pi]*FresnelC[s/Sqrt[Pi]]],
  y -> Function[{s}, Sqrt[Pi]*FresnelS[s/Sqrt[Pi]]]}}
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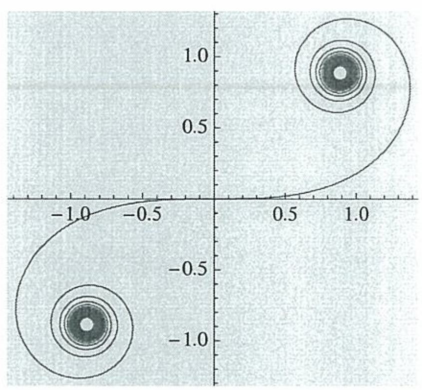

Figure 1. Cornu spiral

2. Partial differential equations (PDEs), which involve more than one independent variable. DSolve can find exact solutions for linear and nonlinear first-order PDEs, and for some higher order equations. For example the Clairaut equation shown below is a nonlinear first-order PDE. If we prescribe the initial values on a parabola in the x-y plane, then DSolve quickly finds a solution for this Cauchy problem.

$$\text{Timing}\left[\text{DSolve}\left[\left\{u(x,y) = x\frac{\partial u(x,y)}{\partial x} + y\frac{\partial u(x,y)}{\partial y} + \frac{1}{2}\left(\left(\frac{\partial u(x,y)}{\partial x}\right)^2 + \left(\frac{\partial u(x,y)}{\partial y}\right)^2\right), u\left(x, -\frac{x^2}{2}\right) = \frac{1}{2}\right\}, u, \{x, y\}\right]\right]$$

$$\left\{0.015, \left\{\left\{u \to \text{Function}\left[\{x, y\}, \frac{1}{2}\left(1 - x^2 - 2y\right)\right]\right\}\right\}\right\}$$

3. Differential - Algebraic equations (DAEs) which typically involving a constraint given in terms of an algebraic equation. DSolve can find exact solutions for linear systems of DAEs with constant coefficients such as the following:

DifferentialEquation = $x'(t) - y(t) == \cos(t)$**; AlgebraicConstraint =** $x(t) + y(t) == 1$**;**

InitialCondition = $x\left(\frac{\pi}{2}\right) == 3$**; DAE = {DifferentialEquation, AlgebraicConstraint, InitialCondition};**

InputForm[sol = DSolve[DAE, {*x*, *y*}, *t*]]

```
{{x -> Function[{t}, (3*E^(Pi/2) + 2*E^t + E^t*Cos[t] + E^t*Sin[t])/(2*E^t)],
  y -> Function[{t}, -(3*E^(Pi/2) + E^t*Cos[t] + E^t*Sin[t])/(2*E^t)]}}
```

As seen in Figure 2, the functions $x(t)$ and $y(t)$ oscillate such that the algebraic constraint (reprsented by a dashed line) is always satisfied:

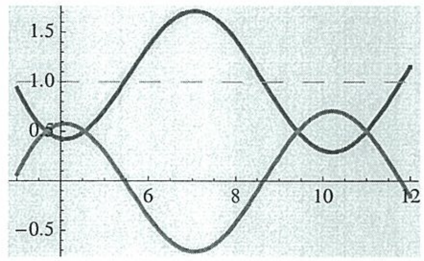

Figure 2. Solution of DAE

The remaining sections contain further details and examples for solving differential equations with DSolve. We will also give some insight into the technology underlying DSolve.

2. Ordinary Differential Equations

Ordinary differential equations (ODEs) may be classified as single equations or systems of equations. The classification of single equations is typically based on the order of the derivatives (first-order, second-order, etc) occurring in the equation and on whether these derivatives and the unknown function occur in a linear or non-linear fashion. We will focus our attention on single equations here.

DSolve can handle all the standard types of first-order ODEs, such as linear, separable variables, Bernoulli, etc. It can also solve Riccati equations which are equations of the type:

$$y'(x) == f(x) + g(x)\, y(x) + h(x)\, y(x)^2$$

The familiar logistic equation can be regarded as a Riccati equation. Here is a solution for this equation along with plots for different values of the parameters (Figure 3):

InputForm[sol = Quiet[DSolve[{ $y'(x) == y(x)\,(1 - b\, y(x))$**,** $y(0) == a$**},** $y(x)$**,** x**]]]**

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{{y[x] -> (a*E^x)/(1 - a*b + a*b*E^x)}}
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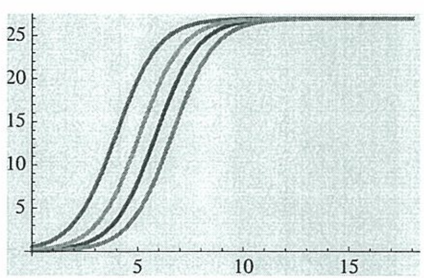

Figure 3. Solutions for the Logistic equation

As with first-order equations, DSolve has algorithms for solving standard types of second-order equations such as those with constant coefficients, Euler-Legendre equations and the equations associated with the classical special functions. Modern methods such as Kovacic's algorithm are used to return solutions for a wide variety of second-order equations with polynomial coefficients. The differential equations which occur in applications often have trigonometric, hyperbolic or exponential coefficients and closed solutions for such equations are found after transforming them to ODEs with polynomial coefficients. For example, the following ODE which describes an aging spring has an exponential function coefficient and is solved in terms of Bessel functions:

$\left(\mathbf{sol = DSolve}\left[\left\{x''(t) + 4\, e^{-\frac{t}{4}} x(t) = 0,\, x(0) = 1,\, x'(0) = 2\right\},\, x,\, t\right]\right)$ **// InputForm**

```
{{x -> Function[{t}, (BesselJ[0, 16*Sqrt[E^(-t/4)]]*BesselY[0, 16] -
      BesselJ[0, 16]*BesselY[0, 16*Sqrt[E^(-t/4)]] + BesselJ[1, 16]*BesselY[0, 16*Sqrt[E^(-t/4)]] -
      BesselJ[0, 16*Sqrt[E^(-t/4)]]*BesselY[1, 16])/(BesselJ[1, 16]*BesselY[0, 16] -
      BesselJ[0, 16]*BesselY[1, 16])]}}
```

A plot of the above solution clearly shows the aging of the spring with increasing time :

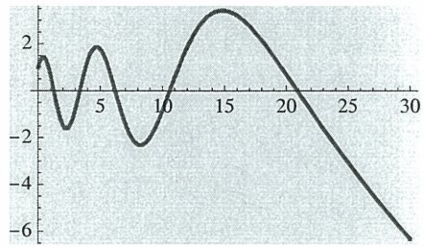

Figure 4. Aging spring

The solution of nonlinear ODEs may require the use of Lie group techniques or even elliptic functions. During the last few decades, the fifty equations discovered by Paul Painlevé (1863-1933) in his classification scheme for second-order nonlinear ODEs have evoked great interest in the scientific community due to their connection with integrable systems. DSolve can find exact solutions for many Painlevé equations such as the following, for which the solution is given in terms of Weierstrassian elliptic functions:

$\mathbf{DSolve}\left[y''(x) = 6\, y(x)^2 + \frac{1}{2},\, y,\, x\right]$

```
{{y → Function[{x}, WeierstrassP[x + C[1], {-1, C[2]}]]}}
```

Finally, equations of order higher than two are often solved by reducing them to second-order or first-order ODEs using factorization methods. For example, the following eleventh-order ODE is solved by recognizing it as the 'tenth symmetric power' of the second-order Airy equation :

$$\mathbf{InputForm}\Big[\mathbf{DSolve}\Big[-256\,(57\,600\,z^3 - 190\,111)\,w'(z)\,z^2 - 3\,669\,600\,w^{(4)}(z)\,z^2 + 16\,368\,w^{(7)}(z)\,z^2 - 1280\,(28\,800\,z^3 - 12\,731)\,w(z)\,z + 176\,(30\,656\,z^3 - 41\,795)\,w^{(3)}(z)\,z + 114\,576\,w^{(6)}(z)\,z - 220\,w^{(9)}(z)\,z + 264\,(122\,624\,z^3 - 13\,995)\,w''(z) - 220\,(2224\,z^3 - 783)\,w^{(5)}(z) - 990\,w^{(8)}(z) + w^{(11)}(z) = 0,\, w,\, z\Big]\Big]$$

```
{{w -> Function[{z}, AiryAi[z]^10*C[1] + AiryAi[z]^9*AiryBi[z]*C[2] + AiryAi[z]^8*AiryBi[z]^2*C[3] +
    AiryAi[z]^7*AiryBi[z]^3*C[4] + AiryAi[z]^6*AiryBi[z]^4*C[5] + AiryAi[z]^5*AiryBi[z]^5*C[6] +
    AiryAi[z]^4*AiryBi[z]^6*C[7] + AiryAi[z]^3*AiryBi[z]^7*C[8] + AiryAi[z]^2*AiryBi[z]^8*C[9] +
    AiryAi[z]*AiryBi[z]^9*C[10] + AiryBi[z]^10*C[11]]}}
```

3. Partial Differential Equations

The solution given by DSolve for a partial differential equation depends on the type of the equation. For a first-order linear PDE, we get the general solution of the equation as seen in the following transport equation example. Here f[1] is an arbitrary function of its argument.

$$\mathbf{SolutionForTransportEquation = DSolve}\left[\frac{\partial u(x, y)}{\partial x} + c\,\frac{\partial u(x, y)}{\partial y} = 0,\, u(x, y),\, \{x, y\},\, \mathbf{GeneratedParameters} \to f\right]$$

```
{{u[x, y] → f[1][-c x + y]}}
```

In the case of a nonlinear first-order PDE, the most useful kind of solution is a complete integral which depends on arbitrary constants such as C[1] and C[2]. For example, the complete integral of the eikonal equation below is a two-parameter family of planes in {x, y, u} space:

EikonalSolution = InputForm[Quiet[DSolve[$\left(\frac{\partial u(x, y)}{\partial x}\right)^2 + \left(\frac{\partial u(x, y)}{\partial y}\right)^2 = 1$, $u(x, y)$, $\{x, y\}$]]]

```
{{u[x, y] -> C[1] + y*C[2] - x*Sqrt[1 - C[2]^2]}, {u[x, y] -> C[1] + y*C[2] + x*Sqrt[1 - C[2]^2]}}
```

The study of travelling waves is a central topic in modern Applied Mathematics since it has applications to the theory of water waves, fibre optics, etc. DSolve can find travelling wave type solutions for nonlinear PDEs such as the Korteweg-deVries (KdV) equation:

KdVSolution = InputForm[Quiet[DSolve[$\frac{\partial u(x, t)}{\partial t} + \frac{\partial^3 u(x, t)}{\partial x^3} + 6 u(x, t) \frac{\partial u(x, t)}{\partial x} = 0$, $u(x, t)$, $\{x, t\}$]] /. $\{c_1 \to 1, c_2 \to -4, c_3 \to \frac{1}{4}\}$]

```
{{u[x, t] -> (12 - 12*Tanh[1/4 - 4*t + x]^2)/6}}
```

By choosing the values of the paramters C[1], C[2], and C[3] carefully one can recreate a graph (and animations) of the soliton observed by John Scott Russell in 1834.

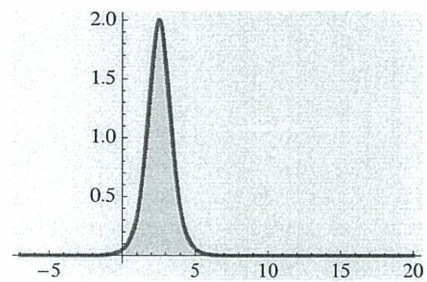

Figure 5. Scott Russell's soliton

The ability to solve PDEs leads to many surprising and beautiful applications such as the generation of combinatorial sequences (Binomial, Stirling, etc.) via the solution of an initial-value problem for a first-order PDE. Further details on this topic are available in Reference 3.

4. The Structure of DSolve

The implementation of DSolve has been done mostly using the *Mathematica* programming language. The internal structure of the function is modular so that new algorithms can be added easily. The modularity also enables DSolve to automatically select the best algorithm for each problem. DSolve makes extensive use of other built-in functions in *Mathematica* such as Integrate and Solve. Typical uses of Integrate are for finding particular solutions of inhomogeneous differential equations and for completing a basis of the general solution while solving homogeneous differential equations. For example, the coefficient of C[1] below is calculated using Kovacic's algorithm while the coefficient of C[2] requires the application of Integrate.

DSolve[$4 x y''(x) + (7 x + 12) y'(x) + 21 y(x) = 0$, y, x]

$$\left\{\left\{y \to \text{Function}\left[\{x\}, e^{-7x/4} C[1] - \frac{e^{-7x/4} C[2] \left(16 e^{7x/4} + 28 e^{7x/4} x - 49 x^2 \text{ExpIntegralEi}\left[\frac{7x}{4}\right]\right)}{32 x^2}\right]\right\}\right\}$$

Together[$\int \frac{e^{\frac{7x}{4}}}{x^3} dx$]

$$\frac{-16 e^{7x/4} - 28 e^{7x/4} x + 49 x^2 \text{ExpIntegralEi}\left[\frac{7x}{4}\right]}{32 x^2}$$

Finally, we remark that the internal structure of DSolve is hierarchical in nature. This means that the solution of complex problems is reduced to the solution of relatively simpler problems, for which a greater variety of methods are available. For example, as seen at the end of Section 2, higher-order ODEs are typically solved by reducing their order to 1 or 2. Thus, a major focus is on development of methods for lower-order differential equations which also occur frequently in Science and Education.

5. Conclusion

The study of differential equations is of fundamental importance in the undergraduate Mathematics curriculum. In particular, the study of symbolic solutions for ordinary and partial differential equations helps in conveying some of the beauty and utility of Mathematics to students. We believe that DSolve, when used in conjunction with the other graphical and numerical capabilities of Mathematica, can be of great assistance in achieving these objectives.

I would like to thank Harry Calkins for a careful reading of this paper.

6. References

[1] P. R. Garabedian, "Partial Differential Equations", AMS Chelsea Publishing, American Mathematical Society, Providence, Rhode Island, 1998.

[2] E. L. Ince, "Ordinary Differential Equations", Dover Publications, Inc., New York, 1956.

[3] D. A. Kapadia, "The Cauchy Problem for first-order partial differential equations", 2006 (to appear in The Mathematica Journal).