EXPLORING INFINITE SERIES WITH TECHNOLOGY

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Abstract
In this paper, we will demonstrate the use of Symbolic Computation Systems, specifically Maple, and Graphing calculators in investigating infinite series. We have selected examples from various areas in undergraduate mathematics where Series play a prominent role and have made a modest attempt to utilize technology to visualize and explore various properties of the Series and calculate or estimate some of the hard-to-determine limits.

Keywords: Maple; Symbolic Computation Systems; graphing calculators, TI 89 infinite Series.

Introduction
We have used technology in our mathematics classes for over fifteen years and we have noticed that many students in undergraduate mathematic courses do not fully appreciate the fine points of infinite series. We believe that the use of technology can help them gain this understanding. The symbolic and computational power of a Symbolic Computation System or a Computer Algebra System (CAS) and hand-held graphing calculators allows us to design experiments, which helps students to overcome some of these challenges. We have chosen most of our examples from topics related to series in undergraduate mathematics as they appear in courses such as calculus, differential equations and foundations of applied mathematics to make our presentation accessible to a larger audience. To show the danger of over reliance on technology, we have also presented examples where Maple, in attempting to calculate the limit of an infinite series, provides inaccurate or misleading results. We have used Maple10 as a representative of CAS mainly because Maple appears to be the software of choice for most of the undergraduate mathematics departments in the US and Canada and we have used the graphing calculator TI-89 because it is the calculator most often used by our students.

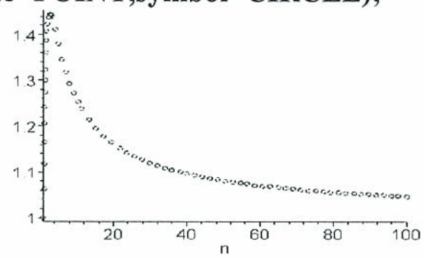
In the following pages we'll present examples to demonstrate our work. For a complete list of the projects, specifically examples on Taylor and Fourier series the reader is encouraged to consult the authors' earlier papers or contact the authors.

1 **Defining, graphing and finding limits of sequences and series**
In the following example, we introduce common Maple commands used for exploring infinite series.

```
> # Sequences
> a:=n->n^(1/n);
```

$$a := n \rightarrow n^{\left(\frac{1}{n}\right)}$$

```
> plot(a(n),n=1..100,style=POINT,symbol=CIRCLE);
```




```
> limit(a(n),n=infinity);
```

$$1$$

```
> # Infinite Series
> s1:=k->sum(1/n^2,n=1..k);
```

$$s1 := k \rightarrow \sum_{n=1}^{k} \frac{1}{n^2}$$

```
> for i from 1 to 5 do;
> evalf(subs(k=10*i,s1(k)));
> end do;
```

1.549767732

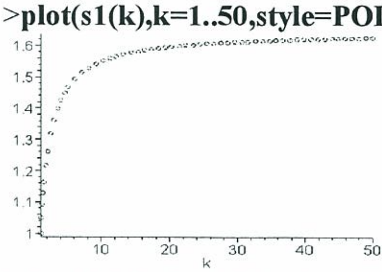
1.596163245

1.612150119

1.620243964

1.625132735

```
>plot(s1(k),k=1..50,style=POINT,symbol=CIRCLE);
```




```
> limit(s1(k),k=infinity);
```

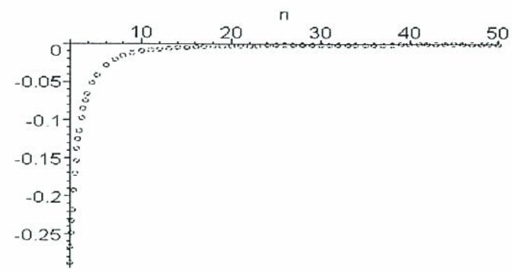
$$\frac{1}{6}\pi^2$$

2 Example of a Telescoping Sum

> **a:=n->ln(1 -(1/n^2));**

$$a := n \to \ln\left(1 - \frac{1}{n^2}\right)$$

➢ **plot(a(n),n=2..50,style=POINT,symbol=CIRCLE);**



> **s1:=k->sum(a(n),n=2..k);**

$$s1 := k \to \sum_{n=2}^{k} \mathrm{a}(n)$$

Let's find the limit of the series and compare to the exact value which is equal to -ln(2)= -0.69314718 [1],

> **limit(s1(k),k=infinity);**

$$-\frac{1}{36} - \frac{3}{2}\ln\left(\frac{3}{4}\right) - 2\,\mathrm{arctanh}\left(\frac{1}{2}\right) - \mathrm{O}\left(\frac{85}{54}\right)$$

> **evalf(%);**

$$-0.6948669580 - 1.\ \mathrm{O}\left(\frac{85}{54}\right)$$

3 Use of Maple in detecting machine errors

Following example, sum of the first 10000 terms of 1/n in a four-decimal machine, demonstrates the use of Maple in detecting machine errors. It is well known that in a forward sum, due to limited number of significant digits, the smaller terms are ignored, whereas in a backward sum the smaller terms add up to larger values and thus generate a more accurate result.

> **Digits:=4;**

$$Digits := 4$$

> **# summing forward (decreasing terms)**

> **s1:=0;**

$$s1 := 0$$

> **for k from 1 to 10000 do s1:=evalf(s1+1/k) end do:**

> **s1;**

$$8.446$$

> **# summing backward (decreasing order)**

> **s2:=0;**

$$s2 := 0$$

> **for k from 10000 by -1 to 1 do s2:=evalf(s2+1/k) end do:**
> **s2;**

$$9.446$$

> **# Find exact value and compare**
> **sexact:=evalf(gamma+Psi(10001),10);**

$$sexact := 9.787606036$$

4 Be careful!

We conclude the Maple examples with a cautionary note about using floating point versus integers. This example which appears in [2] shows a simple change of exponent from an integer "1" to " 1." Creates a totally different and unexpected result. Some might view this as a software bug.

series(1/(1-x)^1,x);

$$1 + x + x^2 + x^3 + x^4 + x^5 + O(x^6)$$

> **series(1/(1-x)^1.,x);**

$$1$$

5 The next example illustrates the abundance of mathematical questions that surround us every day. We urge our students to notice such situations and to use mathematics, and in particular the technology tools available to them, to investigate these interesting occurrences.

A very popular NPR show is Tom and Ray Magliozzi's "Car Talk." The Puzzler feature from the October 23, 2006 show was: You must drive 75 miles. You decide to do the first mile at 75 mph, the second mile at 74 mph, the third mile at 73 mph, … the next to the last mile at 2 mph and the final mile at one mph. ***How long will it take to cover the 75 miles?***

Ray proposed this solution. **RAY:** "What you wind up with is a series of numbers. If you start counting from your destination of one hour, plus a half an hour, plus a third of an hour, plus a fourth, plus a fifth, plus a sixth, and when you're driving at 60 miles an hour, that mile takes you one sixtieth of an hour obviously which is a minute."

RAY Continues: "I got the answer by adding up one plus a half, plus a third, and so on. I converted them all to decimals. There's no discrete answer, the best you can get is an approximation. Here's how you get it: you have 75 terms. You're going to look up the natural log of 75, which happens to be 4.317, and you're going to add it to something called Euler's constant. "

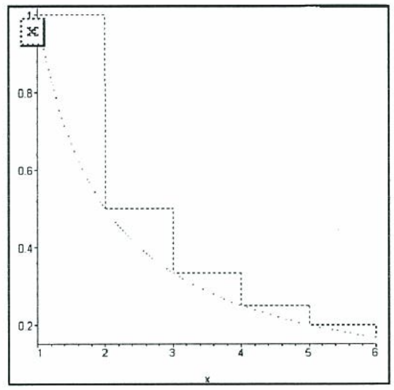
As Ray correctly points out the answer is the sum of 75 rational numbers (1 + ½ + 1/3 + … + 1/75). But, math students know the rational numbers are closed under addition; the answer does not have to be approximated. With Maple or the TI-89 calculator it is simple to find the sum of these numbers. The exact answer is:

$$\frac{67075898176814157144962426221813}{13685172681347672114608764685920}$$

Which we would approximate as

$$4.901355631$$

It will also be interesting to see how easily the student using Maple can penetrate the seemingly strange formula $\sum_{k=1}^{75} \frac{1}{k} = \ln(75) + \gamma$. A graph will help.



The smooth curve is f(x) = 1/x. The rectangles (extended to the x-axis) have area 1, ½, 1/3, … 1/75 (though only a few of them are pictured). Add the areas of the rectangles and you have the problem answer, or, you could calculate the area under the smooth curve and add on the little bits of the rectangles that extend above the curve. The area under the portion of the curve between 1 and 76 is ln(76)[1]. The area of the rectangle pieces above the curve is almost Euler's Constant. Euler's Constant would be the sum of these pieces from 1 to ∞. But we only need part of Euler's Constant; just the part above 1/x between 1 and 76.

- > **int(1/floor(x)-1/x,x=1..76);**
- $$\int_1^{76} \frac{1}{\mathrm{floor}(x)} - \frac{1}{x}\, dx$$
- > **evalf(%);**
- 0.5706222903
- > **evalf(ln(76)+%);**

4.901355630

The above value should be compared with the nine decimal digit representation of the exact value.

We have presented other examples which this paper is too short to contain. It is our hope that what we have written here will inspire you will find opportunities to use technology and take your students through mathematical problems from what they like to refer to as "the real world." It is a great way to engage them and to inspire them to look more deeply into the world using mathematics.

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REFERENCES

[1] Stewart, J., Single Variable Calculus, 5th edition, Thomson publishing, California, 2003.

[2] Abbasian R. and Ionescu A., "*Exploring Some Common Misuses of Maple in Undergraduate* 2003.*Mathematics*", proceedings of Maple workshop, Waterloo, Canada, July 2004.

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[1] Study the diagram carefully and convince yourself that we really need ln(76), not ln(75) as stated on the Puzzler.