EXPLORING SINGULAR DIFFERENTIAL EQUATIONS WITH EXPONENTIAL BOX-SCHEMES

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Computer programs based on exponential box-scheme approximations are developed for solving systems of differential equations that contain small parameters at the highest derivatives or singularities in boundary conditions. The uniform second-order accuracy is obtained for functions and derivatives. The approach is applied to boundary layers with gas injection and combustion.

1. Introduction

Various problems of applied mathematics, thermophysics, and aerodynamics (e.g., stability of mechanical systems and boundary layers [1], supersonic hydrogen combustion [2], and heat protection of hypersonic vehicles [3]) come to solving differential equations with small coefficients at the highest derivatives. The latter leads to the formation of regions with small linear dimensions where gradients of functions are large. The numerical analysis of such problems by traditional box-schemes [4] is limited by non-uniform convergence or even divergence of numerical solutions. In this study, the numerical solutions of the model singular ordinary differential equation [5] have been evaluated for the linear boundary value problem. The developed numerical method is used for the analysis of gas flow parameters in boundary and viscous shock layers under the conditions of blowing on the body surface and nonequilibrium chemical reactions.

From a mathematical point of view, the increase of the flow rate of blowing gas or chemical-reaction rates is equivalent to the existence of a small coefficient at the highest derivative in the boundary-layer (BL) equations [4, 6]. A sublayer with large gradients of functions is created. The gas flow in the boundary layer is studied using a two-point uniform exponential box-scheme and an effective regularization algorithm [7, 8]. The identical problem was considered in [6] by using a three-point exponential box-scheme.

The similar phenomenon is observed in the case of hydrogen combustion at small Reynolds number $Re_0 < 100$ [3, 9]. In this study the models of diffusive combustion of hydrogen, which is injected with different intensity from the surface of a parabolic cylinder into airflow, are considered by using the thin-viscous-shock-layer (TVSL) approach [10] at moderate Reynolds numbers $1500 > Re_0 > 100$.

2. The Model Linear Boundary Value Problem

In general, the method is designed for solving the following model equation:

$$\varepsilon u'' + au' - bu = d \tag{1}$$

Here the parameter ε can accept very small magnitudes, and $a \geq 0$, $b \geq 0$. The solution of the equation (1) with constant coefficients is the following [6]:

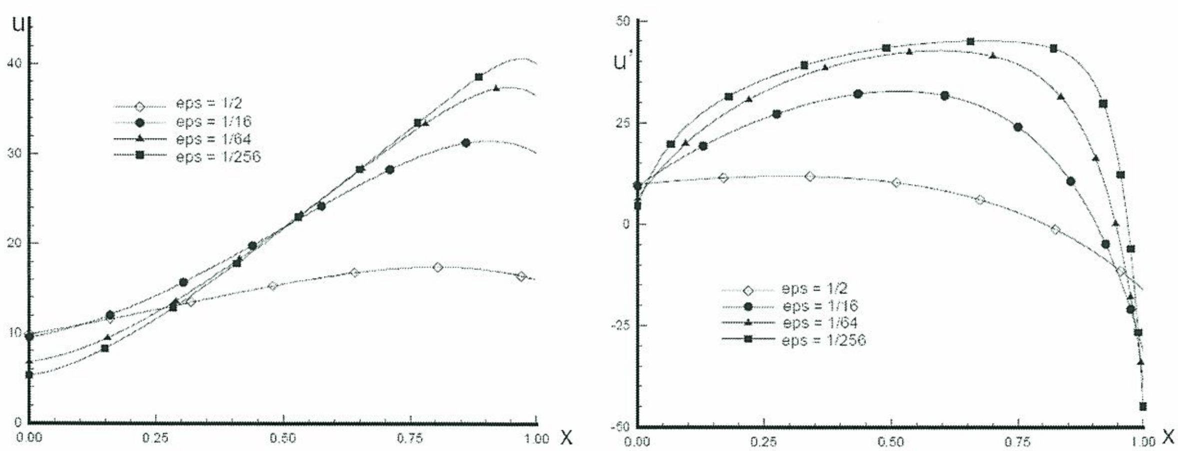
$$u = Ae^{\alpha\eta} + Be^{\gamma\eta} + \psi, \quad \psi = \frac{d}{\varepsilon\alpha\gamma} \tag{2}$$

$$\alpha = -\frac{a}{2\varepsilon} + \sqrt{\frac{a^2}{4\varepsilon^2} + \frac{b}{\varepsilon}} \tag{3}$$

$$\gamma = -\frac{a}{2\varepsilon} - \sqrt{\frac{a^2}{4\varepsilon^2} + \frac{b}{\varepsilon}} \tag{4}$$

where A and B are arbitrary constants, and η is the main variable parameter.

The solution (2-4) is used to obtain the box-scheme characteristics by considering that the functions as well as the derivatives are continuous in the cells [6, 7]. The two-point uniform exponential box-scheme was introduced in [7]. The identical problem was considered in [6] by using a three-point exponential box-scheme. The effective regularization algorithm was developed in [7, 8].



a) Function u b) Function u'

Figure 1. Functions u and u' as solutions of the linear boundary-value problem (5-7).

The linear boundary value problem is studied for the following model singular ordinary differential equation [5] and boundary conditions:

$$\varepsilon u'' - (1 + x^2)u = -(4x^2 - 14x + 4)(1 + x)^2 \tag{5}$$

$$u(0) - u'(0) = 0 \tag{6}$$

$$u(1) + u'(1) = 0 \tag{7}$$

The numerical solutions of the equation (5) with boundary conditions (6) and (7) have been calculated by using the two-point exponential box-scheme [7]. The results for the function u and its derivative u' are shown in Fig. 1. At various parameters ε, they demonstrate the formation of regions with small linear dimensions where gradients of functions are large.

3. Gas Blowing into a Boundary Layer

Consider the perfect-gas flow in the boundary layer (BL) near the stagnation point of a blunt body with uniform blowing at the surface [1]. The system of BL equations acquires the following form [4, 7]:

$$U'' + fU' + \beta(S + 1 - U^2) = 0 \tag{8}$$

$$f' = U' \tag{9}$$

$$S'' + \sigma f S' = 0 \tag{10}$$

where the Faulkner-Scan constant [1] $\beta = (1 + j)^{-1}$ characterizes the pressure gradient in inviscid flow; $j = 0$ or 1 in plane and axisymmetric cases correspondingly; and $\sigma = 0.72$ is the Prandtl number.

Boundary conditions are the following:

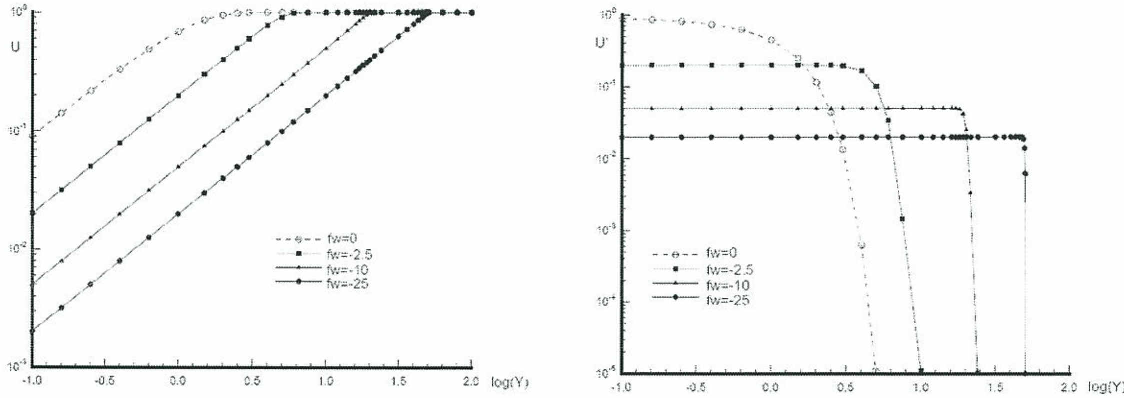
On the body surface ($Y = 0$) considering the gas blowing:

$$f = f_w = \text{const}, \quad U' = 0, S = S_w \tag{11}$$

On the external boundary of the layer ($Y \to \infty$):

$$U = 1, \quad S = 0 \tag{12}$$

In Eq. (11) the parameter f_w characterizes the mass flow rate of the blowing gas. Special box-schemes with uniform convergence [11] or exponential schemes [4, 6, 7] should be used in order to solve the problem at large $|f_w|$. The principal advantages of the two-point box-schemes are: 1) any type of boundary conditions "estimated" accurately [7]; 2) algorithmization of the grid-cell changes is simple [8]; and 3) fluxes of the flow parameters are calculated without additional procedure [11], and the approximation error of the fluxes is the same as that of other terms of the equations. The two-point exponential box-scheme developed [7] has the second order of uniform convergence. The scheme and its regularization algorithm [7] have been used for the numerical solution of Eqs. (8)-(12) under the conditions of moderate and intensive blowing from the thermally isolated body surface ($S_w = 0$). The profiles of the tangential component of the velocity U and its derivative U' along the normal at the stagnation point on the surface of the axisymmetrical blunt body ($\beta = 0.5$) are shown in Fig. 2 for various blowing parameters ($f_w = 0, -2.5, -10, -25$). The presence of the blowing flow significantly changes the flow structure. As the rate of blowing increases, the boundary layer becomes thicker, and the friction on the body surface decreases.

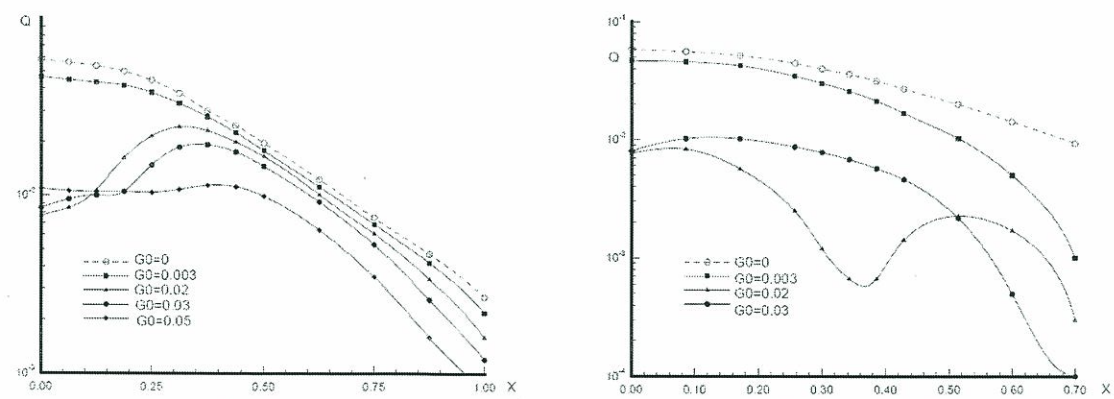


a) Function U b) Function U'

Figure 2. Functions U and U' across the boundary layer for various blowing factors.

4. Combustion in the Thin Viscous Shock Layer

The analysis of hypersonic hydrogen combustion processes in the viscous shock layer near a parabolic cylinder at slot and uniform injections is based on the TVSL equations and boundary conditions [9]. Mathematical description of the nonequilibrium process of hydrogen combustion considered 11 gas components (O_2, N_2, NO, H_2O_2, HO_2, H_2O, OH, H_2, H, O, N) and 35 chemical reactions [12]. An approximation to the equations was constructed using a matrix variant of two-point exponential box-scheme [7, 8, 11]. Modified Newton-Raphson method [9] was used for solving the nonlinear grid equations.



a) Slot injections b) Uniform injections

Figure 3. Heat flux on parabolic cylinder at slot and uniform injections.

The slot injection from the body surface was simulated by using the injection parameter $G = G_0\exp(-\alpha_w/X)$, where G_0 is the value of the parameter at the critical cylinder line, $\alpha_w = 30$, and X is marching coordinate along the surface. Figure 3a presents distributions of the heat flux $Q = 2q/(\rho_\infty V_\infty^2)$ towards the body surface along the coordinate X. The heat from combustion reactions does not prevail over the cooling effect of injection at the considered Reynolds number, $Re_0 = 628$. Even at a low level of injection, the local heat flux decreases. The heat flux is significantly lower than Q at $G_0 = 0$ near the injection zone at moderate injections ($0.02 \leq G_0 \leq 0.05$). A large quantity of injected hydrogen leads to effective cooling of the surface at a large distance from the injection zone at a high level of injection (see diamonds in Fig. 3a). The combustion zone moves from the surface towards the external boundary of the TVSL as the injection intensity increases.

Cooling the porous surface of the parabolic cylinder was modeled at $Re_0 = 628$ and the constant intensity of hydrogen injection, $G = G_0 = const$ along the coordinate X. The heat flux is significantly decreased at uniform injections. For the weak injections (squares in Fig. 3b), the heat flux at the entire computational region is less than Q in the absence of the injection, i.e. $G_0 = 0$. The distribution of the heat flux becomes noticeably distinct at moderate (triangles, $G_0 = 0.02$) and strong (circles, $G_0 = 0.03$) injections, as the result of mutual influence of thermophysical properties of hydrogen, high values of enthalpy of upstream flow, and the presence of combustion zone in the viscous shock layer.

5. Conclusion

The present results demonstrate the effectiveness of using the two-point exponential box-scheme that has a property of uniform second-order convergence in the full range of small parameters such as blowing factors and inverse chemical-reaction rates. It has been found that the moderate blowing leads to decreasing the friction on the surface and the most effective surface cooling occurs at moderate rates of uniform hydrogen injection.

References

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