The Buffon Needle Problem Generalized

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In 1733, George Louis Leclerc, the Comte de Buffon considered the following problem: Given a needle of length l and an infinite set of parallel lines, each d units apart, what is the probability, P(C), that a needle dropped on the lines at random will cross one of the parallel lines [1]. This problem can be used as a simple introduction to the concept of empirical estimation. That fact that it can be used to estimate π makes the problem all the more fascinating.

Using basic calculus and probability [2], [3] it can be shown that $\frac{dN}{2nl}$, where d and l are as above, $l \le d$, n is the number of trials performed, and N is the number of needles which cross a line, is a unbiased estimator of $\frac{1}{\pi}$. Hence $\frac{2nl}{dN}$ can be used to estimate π. (Note: $\frac{2nl}{dN}$ is not an unbiased estimator of π.)

In the remainder of this paper we will look at two generalizations of the needle problem: First we replace the needle with closed-convex regions. Secondly we consider the needle problem on a grid dividing both the x and y axes-this problem is known as the Buffon-Laplace Needle problem.

We shall use the program Flash to create programs to illustrate the problems discussed above.

Where l and d are defined above, it can be shown [3] that P(C) is given by:

$$P(C)=\begin{cases}\dfrac{2}{\pi}x & x\le 1\\[2mm] \dfrac{2}{\pi}\left[x-\sqrt{x^2-1}-\sin^{-1}\left(\dfrac{1}{x}\right)\right]+1 & x>1\end{cases}$$

$$\text{where } x=\frac{l}{d}$$

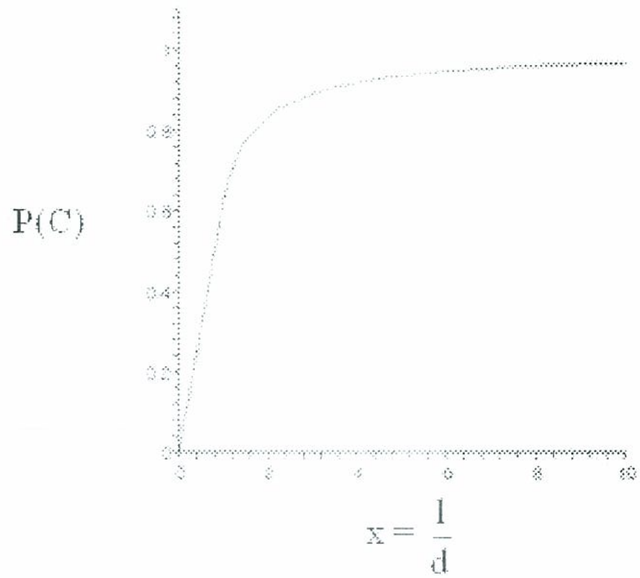

Figure 1

The following is a typical run where n=1000 and l=d=100. As the number of crossing was N=628, the estimate of π is given by $\frac{2}{\pi}=\frac{628}{1000}\Rightarrow \pi$ is approximated by 3.1847.

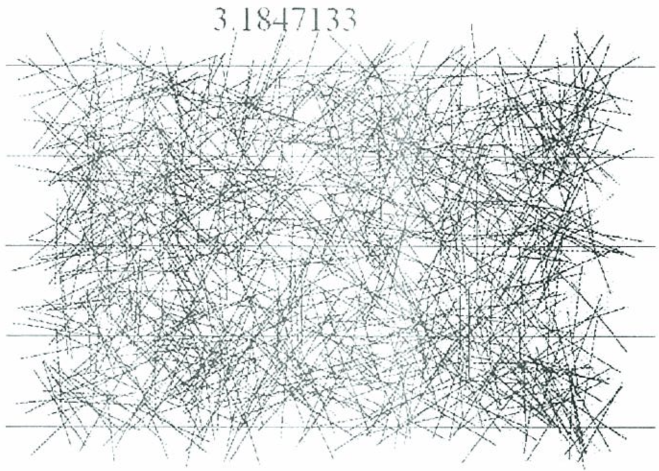

Figure 2

Replacing the Needle with Closed-Convex Regions

Let R be a closed-convex region with generalized diameter $\leq$ d (the greatest distance between two points on the boundary $\leq$ d [4]) and perimeter=a.

Then the probability that R dropped on the grid of parallel lines will intersect at least one of the lines is given by [3], [5]:

$$P(C) = \frac{a}{\pi d}$$

By way of example: Given a rectangle=75x30, n=500, and d=100

$$P(C) = \frac{a}{\pi d} = \frac{210}{\pi \cdot 100} = .662$$

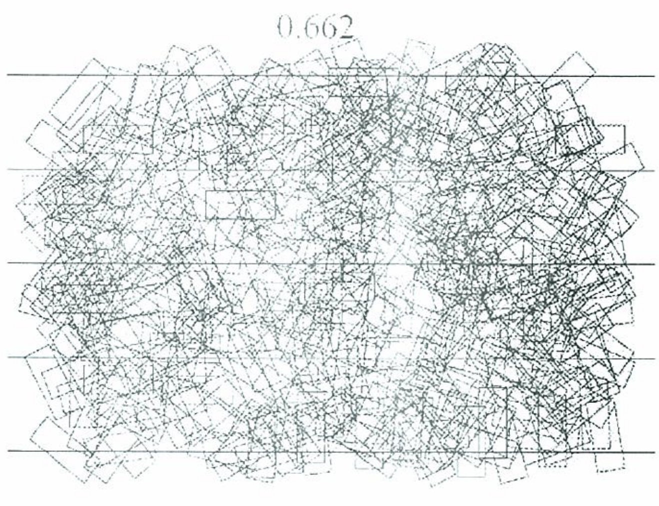

Figure 3

Giving us an estimate of $\dfrac{1}{.662 * \left(\dfrac{100}{210}\right)} = 3.1722$ for π.

The first two examples show us that the Buffon needle method is not a good way to approximate π efficiently. Even using 10,000 needles often gives an error of approximation of over .01.

Buffon-Laplace Needle Problem

Another generalization of the Buffon needle problem is to create a grid of lines parallel to the x and y axes. The distance between each of the lines parallel to the x-axis is defined by a and the distance between each of the lines parallel to the y-axis is defined by b. Needles of length $l \leq \max\{a, b\}$ are then dropped on the gird.

Where P(C) is again the probability that a needle hits any line in the grid, it can be shown that [6], [7], [8]:

$$P(C) = \frac{2l}{\pi a} + \frac{2l}{\pi \cdot b} - \frac{l^2}{\pi ab}$$

In a run of n=1000 needles with a=b=l=100 we saw that P(C)=.955.

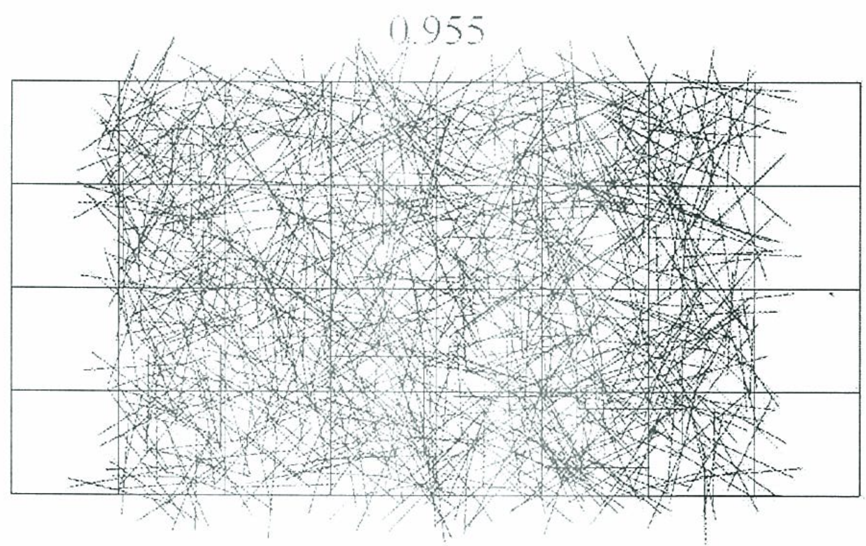

Figure 4

Which gives us an estimate of

$$\frac{2\cdot 100}{\pi\cdot 100}+\frac{2\cdot 100}{\pi\cdot 100}-\frac{100^2}{\pi\cdot 100\cdot 100}=\frac{3}{\pi}=.955$$

Or $\pi \doteq \frac{3}{.95} = 3.1579$.

Circular Arguments

In writing programs to illustrate the Buffon Needle Problem we need to have the angle that a needle makes with a line parallel to the x-axes have a uniform distribution on $[0, \pi]$. This makes use of an estimate of π-which is what we are trying to estimate-a circular argument.

A possible way out of this problem is to use the fact that as π is irrational i=1,2, 3,... give a uniform and dense distribution on the unit circle [9]. So to compute the angle of the needles: precompute sin(1) and cos(1) and use [10]

$$\sin(i+1)=\sin(i)\cos(1)+\cos(i)\sin(1)$$
$$\cos(i+1)=\cos(i)\cos(1)-\sin(i)\sin(1)$$

Use high precision to avoid round-off errors

References:

[1] *Buffon's Noodle Problem*, Mathematical Notes, October, 1969, J. F. Randley, p. 916-918.

[2] *Buffon's Needle Problem*, http://www.ds.unifi.it/VL/VL_EN/buffon/buffon2.html.

[3] *Buffon's Needle Problem*, http://mathworld.wolfram.com/BuffonsNeedleProblem.html.

[4] *Generalized Diameter*, http://mathworld.wolfram.com/GeneralizedDiameter.html.

[5] G. Gnedensko *The Theory of Probability*, 2nd Edition Chelsea 1963.

[6] *Buffon-Laplace Needle Problem*, http://mathworld.wolfram.com/Buffon-LaplaceNeedleProblem.html.

[7] H. Solomon, *Geometric Probability*, SIAM, p.3-6, 1978.

[8] J. V. Uspensky *Laplace's Problems-Introduction to Mathematical Probability*, New-York: McGraw-Hill, p. 255-257, 1937.

[9] Hardy, GF and Wright, EM, *An Introduction to the Theory of Numbers*, 5th Edition, Oxford University Press, 1980.

[10] Brodie SE, *Buffon's Needle Problem*, http://www.cut-the-knot.org/fta/Buffon/buffon9.shtml.