LINEAR ALGEBRA AND LINEAR PROGRAMMING
IN A FINITE MATHEMATICS COURSE

Peter Staab
Fitchburg State College
160 Pearl St. Fitchburg, MA 01420
pstaab@fsc.edu

1 Introduction

The Finite Mathematics course at Fitchburg State College is a course mainly taken by students that need a mathematics course for graduation. About half of the course is topics in linear algebra including linear programming and many topics require matrix row operations. The technique for solving many linear algebra problems involves only arithmetic, however students weak in these skills do not understand how to solve them and for most students, the technique of row reduction is difficult.

I have developed WebCAS, a series of educational mathematical tools. This includes a simple-to-use tool that allows students to enter in a matrix and perform standard row operations in a straightforward manner using the notation they experience in the classroom. I use this tool to teach students the concepts of row operations and explain the process of solving linear systems using Gaussian or Gauss-Jordon Elimination and also for finding inverse matrices. It also lends itself nicely to solving linear programming problems using the Simplex Method of Dantzig.

2 Linear Algebra Topics in the Finite Mathematics Classroom

Finite Mathematics at Fitchburg State is taken mainly by students needing to pass a math class for graduation requirements. Generally, these students are not strong in basic mathematics, including arithmetic and algebra. Finite mathematics is a great course for such students, because there is some interesting mathematics that can be done without strong skills. The following is a list of relevent topics generally covered in this course:

- Lines
- Linear Systems
- Gaussian Elimination*
- Matrices and Matrix Operations*
- Linear Programming
- The Simplex Method*

These topics cover between one-half and two-thirds of a typical finite mathematics course. References [2] and [1] are two texts that have been used. The topics with asterisks above heavily use row operations on matrices and I have found that many students struggle with this and thus do not grasp the ideas behind many of these topics and often cannot solve a non-trivial problem.

Without using any software within the class, typically one can only cover Gaussian elimination with systems of 2 variables and 3 variables, and in most cases the matrices require a simple solution path.

2.1 The Simplex Method

The simplex method is a technique developed by Dantzig in the 1940s to solve linear programming problems[1]. This is a nice topic for this class for a variety of reasons. First, the simplex method is used throughout the business community to solve difficult problems involving scheduling and production. Thus it is quite practical. Secondly, it is a relatively new mathematical technique; in most classes we discuss mathematics that is hundreds of years old.

Linear programming has a vast number of practical problems and providing such motivation convinces the students of its importance. I generally introduce the subject as a personal example: *There are a number of things that you must spend time on in your life: classes, work, family. There are constraints (hours in a day, things your are required to attend, places you must go).* I then construct a simple problem related to this.

Initially in this course, the geometry of linear programming problems[2] is discussed, then the simplex method[1, 2] in introduced. A simple example of such a problem is to maximize $f(x, y) = 2x + 3y$ subject to the following constraints:

$$\begin{cases} x + y \leq 20, \\ x + 4y \leq 68, \\ 5x + 2y \leq 70, \\ x \geq 0, \ y \geq 0. \end{cases}$$

And the initial step is to create the corresponding simplex matrix is:

$$\begin{array}{cccccc|c} x & y & u & v & w & M & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 20 \\ 5 & 2 & 0 & 1 & 0 & 0 & 70 \\ 1 & 4 & 0 & 0 & 1 & 0 & 68 \\ \hline -2 & -3 & 0 & 0 & 0 & 1 & 0 \end{array} \quad (1)$$

where u, v, w, M are slack variables. As it can be seen, for a relatively simple linear programming problem (2 variables and 3 constraints), the simplex matrix is quite large (4 rows by 7 columns). This is quite intimidating to students whose largest matrices solved by hand are 3 by 4. And again for students a little shaky with row operations, the simplex method is quite daunting. Finding the solution requires pivoting about two or three elements of the matrix, requiring in total about 10 row operations in total. And this is for the simplest of problems.

[1] A linear programming problem is an optimization of a linear function with linear constraints that form a feasible region. A simple theorem shows that if the region is bounded, then the solution occurs at a vertex of the region. The simplex method uses a matrix to traverse the vertices in an efficient manner.

[2] I intend to develop a WebCAS tool to present and solve these problems as well in the future. The students in this class have difficulty with plotting linear functions and finding the appropriate feasible region as well.

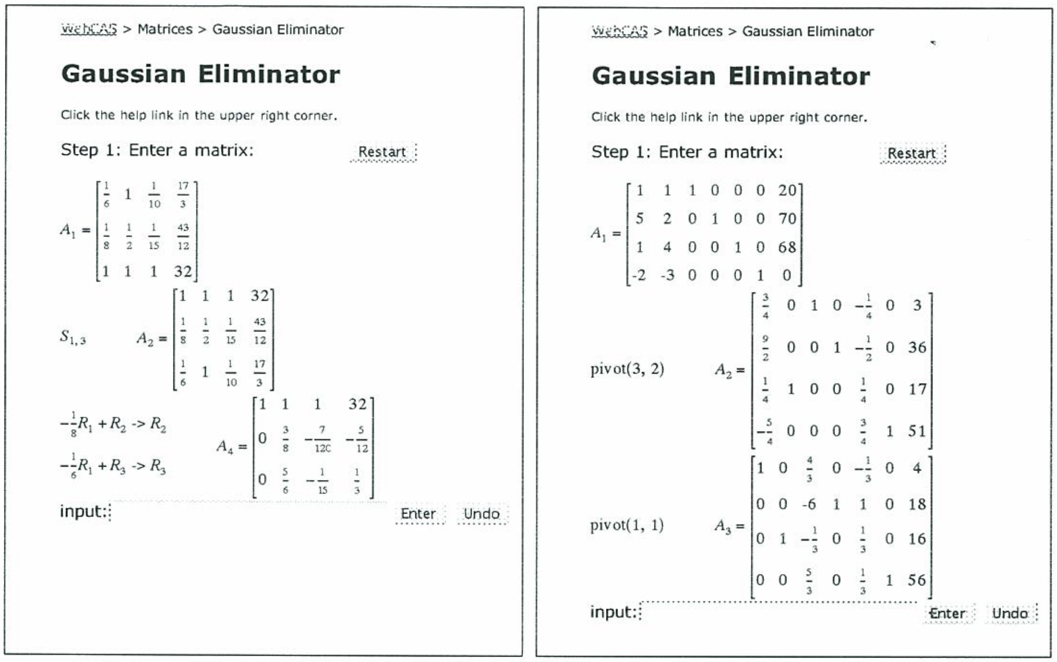

Figure 1: Two examples of use of the Gaussian Eliminator tool in WebCAS. The figure on the left shows a few steps of row reductions as in the method of Gaussian elimination. Once a matrix is entered, row operations are typed in in much the same way as it would be hand-written. Note also, that the Gaussian Eliminator uses rational numbers and simplification of such numbers is automatically done. MathML is used to render the matrix and the fractions. The figure on the right is the same tool used to solve a linear programming problem via the simplex method. The steps shown use the built-in command `pivot` to perform the given operations.

When I have taught the simplex method in the past without use of computer software, I find that only the best students grasp the method and rarely does anyone get through an entire problem without making a mistake along the way.

3 Row Operations on Matrices using WebCAS

WebCAS[3] is a set of mathematical tools designed for students that is accessed within a web browser[3]. Information on WebCAS and links to the individual tools can be found at `http://falcon.fsc.edu/pstaab/webcas`. Figure 1 shows two examples of using the *Gaussian Eliminator* tool of WebCAS within the Firefox browser. The *Gaussian Eliminator* tool performs the three standard row operations of 1) swapping two rows; 2) multiplying a row by a nonzero constant; and 3) multiplying a row by a constant and adding to a multiple of another row.

[3]The name arises from Web (since it is accessed from a web browser) and CAS or Computer Algebra System, the technical term for software that performs many mathematical operations in a symbolic fashion.

With the *Gaussian Eliminator* tool of WebCAS, students are forced to think about the technique of Gaussian elimination. Students who are weak in arithmetic struggle through this method mainly because one small mistake results in an unexpected result with more-difficult arithmetic operations along the way. The *Gaussian Eliminator* puts the impetus on the row operations and not on the arithmetic and allows the students to concentrate on the method.

The figure on the left side of Figure 1 shows a few steps of Gaussian elimination. The original matrix is entered with integers and rational numbers, which is natural because it is the way it is solved by hand. The first step is a row swap which is entered as `S13`, nearly the same as is written. The next two row operations are sums of multiples of rows. These are entered as `(-1/8)R1+R2->R2, (-1/6)R1+R3->R3` and both can be performed in a single step. Again, entering the row operation is nearly identical to written work and when possible, two or more operations can often be done simultaneously. This is precisely how I solve this problem by hand. Because of the format of the input, this should enhance the student's written work as well.

The figure on the right side of Figure 1 shows the solution to the above linear programming problem in (1) using the *Gaussian Eliminator*. First, notice that there is a pivot method that is built in. Generally, when presented, I have a student go through the individual operations involved in pivoting, then also learn the simplex method using the built-in `pivot` command. When this is used in the classroom, one can concentrate on the steps involved in the simplex method which include choosing the appropriate column and row for pivoting.

3.1 Projects using WebCAS

One advantage to using WebCAS is to enhance the understanding of the techniques in row operations and the simplex method. A secondary reason for using the tool is to using projects in class. Currently I only have two projects in the Finite Math course. The first project has the students solve standard textbook-type problems in solving linear systems, Matrix Operations and finding inverses using Gauss-Jordon. The goal is to familiarize them with the tool.

The second project is to get the students to solve a large linear programming problem using the simplex method. The students will create each problem themselves including setting up the objective function, linear constraints and solving with the simplex method. And as they use WebCAS, they can focus on the solution technique, and can forgo the tedious arithmetic found in solving the problem.

4 Pedagogy and WebCAS

As an educator, I think it is important to see hand-written material from students. Such work reveals the student's thinking and true understanding. It is similar to an essay in many non-scientific fields in that when a student is presenting all supporting material for a problem (or essay topic), the teacher can determine if she truly grasps the material. When used in conjunction with hand-written homework, the *Gaussian Eliminator* should improve their understand and mathematical writing skills.

As I designed the Gaussian Eliminator and other tools in WebCAS, I had important goals in mind.

- Students should understand the method they are using.
- The tools should enhance their understanding.
- It should be natural to use if they understand the mathematics.
- It should be powerful enough to handle small *real-world* problems.

5 The Technology

This is a brief discussion of the technology that is used to create the tools in WebCAS. There are four basic parts of this that work in conjunction: XHTML, javascript, SVG and MathML. Most web browsers support parts of each of these technologies, however the most robust platform has been the Firefox web browser, which natively supports most aspects of these technologies on the standard three computer platforms.

- XHTML is a more modern version of HTML, the basic language of the world wide web.
- javascript is a scripting computer language that most web browsers use to create interactive web pages.
- SVG stands for Scalar Vector Graphics and is used to create many kinds graphics within a web page.
- MathML is a markup language to write mathematics. Most mathematical software packages support this and WebCAS uses this to format any mathematical output.

It's important to emphasize that because these tools using both integers or rational numbers for their standard number type instead of floating points. This allows the matrix operations to be accomplished in a way similar to that in class, and students do not need to worry about problems with round-off in floating point numbers.[4] (The WebCAS tools accept floating point numbers, but I emphasize that students should stick to integers and rational numbers.)

References

[1] Larry J. Goldstein, David I. Schneider, and Martha J. Siegel, *Finite Mathematics and its Applications*, Pearson Education, 2007.

[2] Margaret L. Lial, Raymond N. Greenwell, and Nathan P. Ritchey, *Finite Mathematics*, Pearson Education, 2005.

[3] Peter Staab, *Webcas: Mathematical teaching tools on the web*, A short webpage introducing the mathematical web tools., `http://falcon.fsc.edu/pstaab/webcas`.

[4] This is standard in many CAS programs such as Maple and Mathematica, but is rare in general computer programs.