MODELING WITH DISCRETE DYNAMICAL SYSTEMS

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Modeling with Discrete Dynamical Systems (DDS) is a powerful modeling tool. It enables students to think about the issue of correctly modeling a situation. The functional notation allows students to better understand the concept of the function. All our solutions are illustrated and solved numerically and graphically using a TI-83 Plus. The difference equations as models involving prescribed drug dosages, non-linear population growth, and a system involving competitive hunters are illustrated in this paper. These models are robust in that they can be used in many different scenarios and disciplines. Our analysis develops the concepts of equilibrium and stability, which are easily seen from the numerical and graphical output.

Model a prescribed drug dosage.

One time drug: Novocain (Procainamide Hydrochloride), injected as an anesthetic for minor surgical and dental procedures, is eliminated from the body primarily by the kidneys. Loosely speaking, during any 1-hour period, the kidneys take a fixed percentage of the blood and remove medicine from the blood. Let's assume the kidneys purify 1/5 of the blood every one-hour period.

Let $u(n)$ = the amount of the prescribed drug in our system after n (one-hour) periods.
Then, $u(n+1) = u(n) - .20\, u(n) = .80\, u(n)$, $n = 0, 1, 2, 3, \ldots$
Let's assume we are given 500 mg of the drug in period 0, so $u(0)=500$.

Let's iterate and graph the DDS to see what happens over a long period of time. In this hand-out we demonstrate how a TI-83 Plus can be used but you could also use an EXCEL spreadsheet. This model is a discrete dynamical system, DDS.

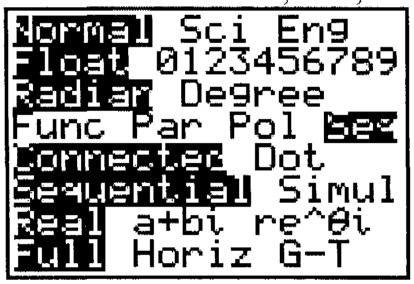
Some background is required. Using a DDS involved modeling with the paradigm:

Future = Present + Change

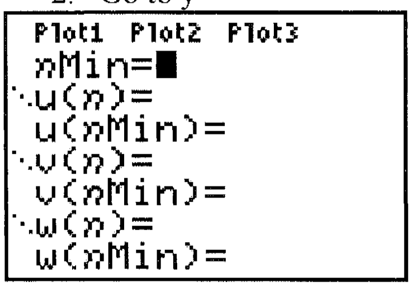
A DDS is a discrete function that can be used to model many situations, such as mortgage of a home, car financing, investment or financial alternatives, prescribed drug dosages, population dynamics such as predator-prey problems, competitive hunter problems, and genetics. All these examples are covered in this article, but there are many more. Discrete dynamical system models are some of the more robust models known. A DDS might sound familiar from calculus which is the study of change. In calculus our paradigm is changed to ***change = future value - present value*** which is known as a difference equation.

Now, let us return to our example. We can build the solution on the TI-83 calculator.

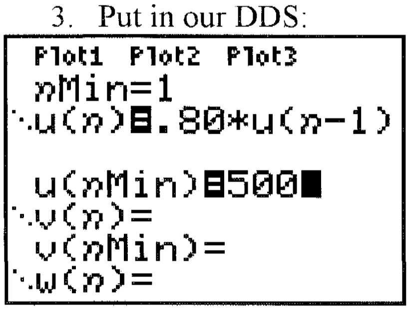
1. Go to **MODE**, func, **SEQ.** (2[nd] quit)



2. Go to y=

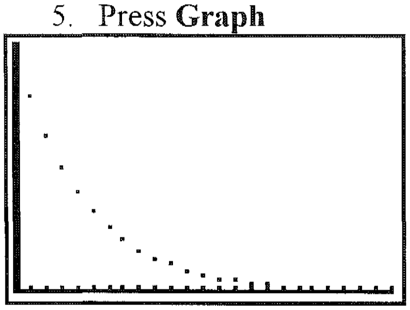


3. Put in our DDS:



4. Set the window for n [0,24], x [0,24], y [0,505]

5. Press **Graph**



We clearly see that this drug decays over time. There is less and less of the drug in our system after each hour. Remember that tingling feeling as the feeling begins to return to your face. Hopefully, we received this drug (via injection) only one time.

More Than One Time Drug Dosage

Now, let's consider a drug taken more often than once. *CIPRO* is a drug for combating many infections, including anthrax. Let's assume that during a one-hour period that our kidneys purify 1/4 of this drug from the blood. Let's assume that the dosage is 16 mg each time period. Let's see what happens between each dosage.

Write the mathematical model that represents this system. Our model is $d(n+1) = .75\, d(n)$, $d(0)=16$ *mg*, $n=0,1,2,3,\ldots$ *(in one-hr periods)*. Let's assume that to be effective, you must have at least 6.75 mg of the medicine in your blood. How often should you take the medicine?

Now, let's remodel assuming that every 4-hour period we take 16 mg of the drug and that we don't have any in our system when we begin, $d(0)=0$. Let's assume that in a 4-hour period the kidney's only purify 60% of the drug. The model is $d(n+1)=d(n)-.6*d(n) + 16$ or $d(n+1) = .40 * d(n) + 16$ for $n=0,1,2,3,\ldots$ where n now represents 4-hour periods.

What happens now? Look at a long period of use for this drug. Describe what you see from the graphical output. This shows a *__stable equilibrium__* value. The definition of an equilibrium value is when $d(n+1)=d(n)$, *INPUT=OUTPUT,* and there is no change. This equilibrium value is _____mg of this drug. We will talk about the "stable" part later.

__Car Finance Example__

You want to buy a $20,000 new car and you can afford a monthly payment of only $400 per month. As you shop around you find a dealer that will give you $1,500 cash back if used as a down payment and 6.9% per year financing compounded monthly. Can you get your new car?

Let a(n) = the amount that you owe the financing company after n months

$a(n+1)=a(n)+(.069/12)\, a(n) - 400$

$a(n+1) = (1 + .069/12)\, a(n) - 400$

How can we solve this DDS? Well, one method is by numerical iteration. We know that a(0)=$18,500. Go ahead and work through this problem.

__Nonlinear Discrete Dynamical Systems__

In this section we build nonlinear discrete dynamical systems to describe the change in behavior of the quantities we study. To remind us let's define a nonlinear DDS-- If the function of $a(n)$ involves powers of $a(n)$ (like $a^2(n)$), or a functional relationship (like $a(n)/a(n-1)$), we will say that the discrete dynamical system is **nonlinear**.

__Growth of a Yeast Culture__

We often model population growth by assuming that the change in population is directly proportional to the current size of the given population. This produces a simple, first order DDS similar to those seen before. It might appear reasonable at first examination, but the long-term behavior of growth without bound is disturbing. Why would growth without bound of a yeast culture in a jar (or controlled space) be alarming?

There are certain factors that affect population growth, such as resources like food, oxygen, and space. These resources can support some maximum population. As this number is approached, the change (or growth rate) should decrease and the population should never exceed its resource supported amount.

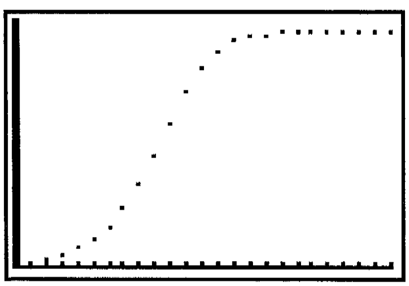
Problem: Predict the growth of yeast in a controlled environment as a function of the resources available and the current population. We assume that the population size is best described by the weight of the biomass of the culture. We define y(n) as the population size of the yeast culture after period n. There exists a maximum carry capacity, M, that is sustainable by the resources available. The yeast culture is growing under the conditions established.

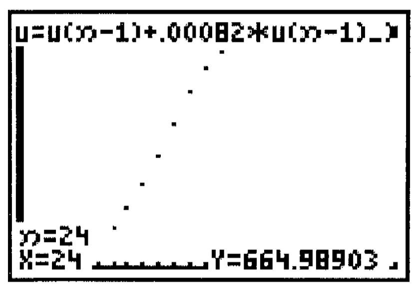
Model: $y(n+1) = y(n) + k\,y(n)\,(M-y(n))$ where

k is the growth rate of the culture

M is the carrying capacity of our system.

In our experiment, we find by data collection that the growth rate, k, is approximately 0.00082 and the carrying capacity of the biomass is 665. This model is $y(n+1) = y(n) + .00082\ y(n)\ (665-y(n))$. Again, this is nonlinear because of the $y^2(n)$ term. Using an initial condition, biomass, of 9.6 the model shows stability in that the population (biomass) of the yeast culture approaches 665 as n gets large. Thus, the population is eventually stable at approximately 665 units. Try this on your calculator



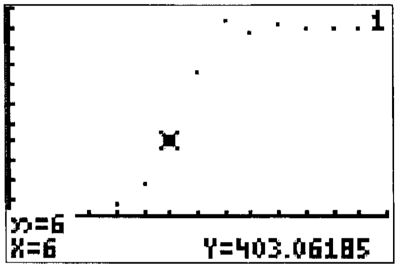


Spread of a Contagious Disease

There are 1000 students in a college dormitory and some students have been diagnosed with meningitis, a highly contagious disease. The health center wants to build a model to determine how fast the disease will spread.

Problem: Predict the number of students affected with meningitis as a function of time. Let m(n) be the number of students affected with meningitis after n days. We assume all students are susceptible to the disease. The possible interactions of infected and susceptible students are proportional to their product (as an interaction term).

The model is $m(n+1) - m(n) = k\ m(n)\ (1000-m(n))$ or $m(n+1) = m(n) + k\ m(n)\ (1000-m(n))$. It is found that two students returned from spring break with meningitis. The rate of spreading per day is characterized by k=0.0015. It is assumed that a vaccine can be in place and students vaccinated within 1-2 weeks.



The results clearly show that most students will be affected within 2 weeks. Since more than 40% will be affected within six days, every effort must be made to get the students vaccinated sooner.

Systems of Dynamical Systems

Now let's consider systems of difference equations (DDS). For selected initial conditions, we build numerical solutions to get a sense of long term behavior of the system. We can study the resulting pattern of the numerical solutions, even if the problem is nonlinear like the following:

Competitive Hunter Problem

Competitive hunter models involve species vying for the same resources (such as food or living space) within their habitat. The effect of the presence of a second species diminishes the growth rate of the first species. We now consider a specific example concerning trout and bass in a small pond. Hugh Ketchum owns a small pond that he uses to stock fish and eventually plans to allow fishing. He has decided to stock both bass and trout. The fish and game warden tells Hugh that after inspecting his pond for environmental conditions he has a solid pond for growth of his fish. In isolation, bass grow at a rate of 20% and trout at a rate of 30%. The warden tells Hugh that the interactions for the food affect trout more than bass. They estimate the interaction affecting bass is 0.0010* bass*trout and for trout is 0.0020* bass*trout.

Model: Let B(n) = the number of bass in the pond after period n, T(n) = the number of trout in the pond after period n, and B(n) T(n) = interaction of the two species.

$$B(n+1) = 1.20\, B(n) - 0.0010\, B(n)\, T(n)$$
$$T(n+1) = 1.30\, T(n) - 0.0020\, B(n)\, T(n)$$

Hugh initially buys 151 bass and 199 trout for his pond. Are there equilibrium values for the bass and trout? Are they stable? Can the bass and trout coexist?

Conclusion

In these examples, we presented the models as our main focus. Although there are many different examples to chose from, we presented only a few in this limited space. The concept of equilibrium, an important concept, in DDS is defined as when Input=Output or when change stops. This is seen numerically as the sequence converges. The concept of stability is also very important. An equilibrium value is stable if when we begin iterating a DDS from any point near the equilibrium value, we always converge to the equilibrium value; otherwise, the equilibrium is not stable. Discrete dynamical systems can be used in courses from college algebra through advanced mathematics electives.