RUMORS AND FLU AND NEW PRODUCTS, OH MY!
LOGISTIC MODELING TO THE RESCUE!

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What do rumors, flu, and new products have in common? Data on these phenomena often can be fit to logistic growth models. Our workshop focused on exploring the mathematics of the logistic growth function, and using the capabilities of the TI-84+SE to fit logistic growth models to flu and rumor data. The purpose of this paper is to examine new product data, fit a logistic growth model, if appropriate, to the data, and pose/answer some questions about the data using the model.

The logistic growth model is an example of limited growth. In general, it is given by $Y = \frac{L}{1+ae^{-bx}}$ and is obtained from solving the differential equation $\frac{dY}{dX} = kY(L-Y)$ where L, k ,a, b >0. The constant L represents the carrying capacity. Instructors may wish to have students investigate the role and impact of L, a, and b on the graph of a logistic equation. The characteristic S-shape of the basic logistic growth model $Y = \frac{1}{1+e^{-x}}$ may be seen in Figure 1

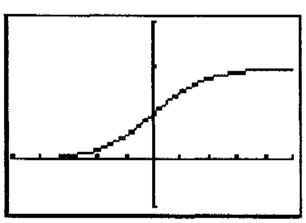
Figure 1: Basic logistic growth model

The data we will consider is U.S. DVD Hardware Sales for each quarter from 1997-2005. In the table below, data are in millions of units; the source is www.dvdinformation.com/industryData/index.cfm.

Quarter	1997	1998	1999	2000	2001	2002	2003	2004	2005
1st Q	0.030	0.094	0.358	1.350	2.220	3.565	4.858	6.855	7.741
2nd Q	0.079	0.149	0.611	1.435	2.404	3.750	5.506	6.057	6.006
3rd Q	0.077	0.244	0.880	1.550	2.537	4.740	6.470	6.593	6.250
4th Q	0.119	0.459	1.701	5.542	9.501	13.058	16.90	17.621	16.740
Yearly Total	0.305	0.946	3.55	9.877	16.662	25.113	33.734	37.126	36.737

Figure 2: U.S. DVD Hardware Sales in millions of units

From the table in Figure 2, we will investigate the yearly total over time by first entering these data using the STAT features of the TI-84+SE. Figure 3 displays Year in L1 (0 corresponds to 1997) and Yearly Total in L2. Figure 4 shows the resulting scatter plot using ZOOMSTAT.

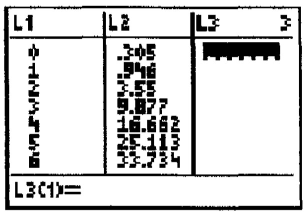

Figure 3: Yearly DVD hardware sales for 1997-2005

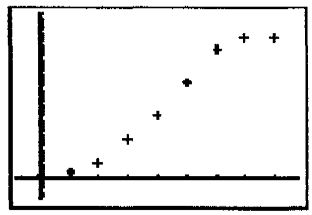

Figure 4: Scatter plot of yearly DVD hardware sales

The data appear to conform to the shape of a logistic growth function. Next, we will use the logistic regression features of the TI-84+SE to obtain a logistic model for the DVD hardware sales data. Values for the logistic model are set to report to two decimal places.

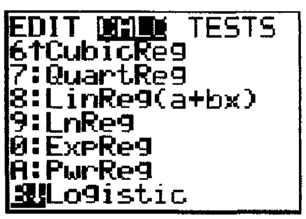

Figure 5: logistic regression selected

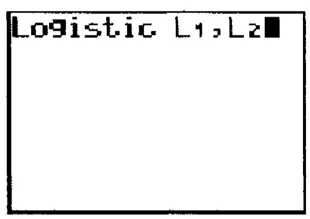

Figure 6: L1 is year; L2 is DVD hardware sales

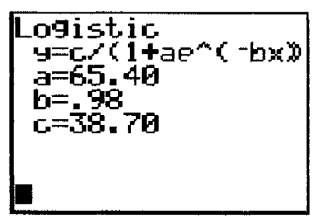

Figure 7: regression results

Our next step is to paste the regression results in the Y= editor and obtain a graph of the logistic model along with the scatter plot. The following figures display this segment of our analysis.

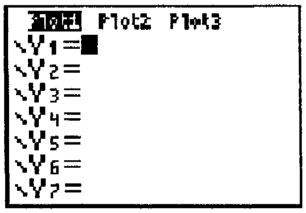

Figure 8: Y1 in the Y=editor

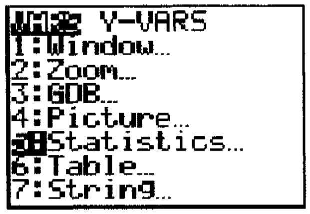

Figure 9: Select Stats

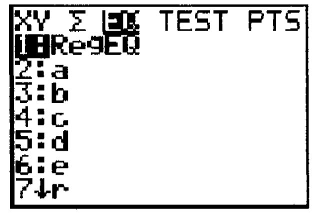

Figure 10: Select regression equation

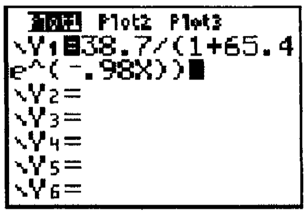

Figure 11: regression equation pasted in Y1

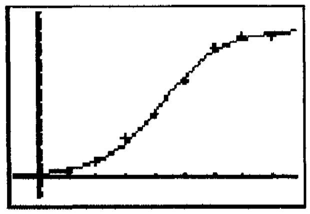

Figure 12: Graph of scatter plot and regression equation

At this point, we can now generate and answer various questions based on the logistic model.

1. According to the model, approximately what is the ceiling or leveling off value or carrying capacity on the maximum number for U.S. DVD hardware sales to the nearest whole number? The following screens show that DVD hardware sales of about 39 million units is a possibility based on the model.

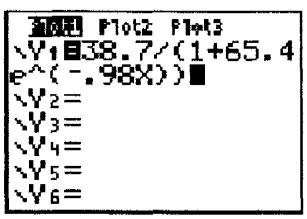

Figure 13: regression equation

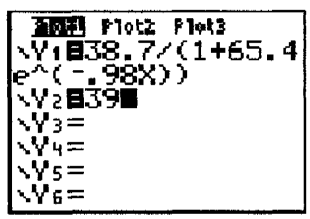

Figure 14: carrying capacity in Y2

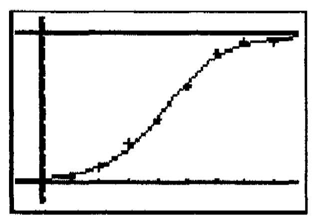

Figure 15: graph of model and carrying capacity

2. According to the model, about when are U.S. DVD hardware sales growing the fastest? What is this rate? Since this question involves the first derivative, the instructor may wish to have students answer this question analytically. Another option would be to compare by-hand analytical with TI numerical approximation results. Here, however, the TI is used to obtain all results. Note that the scatter plot has been turned off and the regression equation in Y1 has been deselected so that only the graph of its first derivative in Y2 is displayed. A slight adjustment was made in YMAX in the WINDOW after the graph was obtained to leave some room to clearly see the displays in Figures 19, 21 and 22. From Figure 22, we see that DVD hardware sales are growing the fastest around the beginning of the second quarter in 2001 at a rate of about 9.48 million units per year.

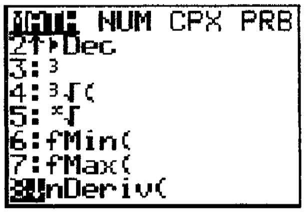

Figure 16: Select numerical derivative

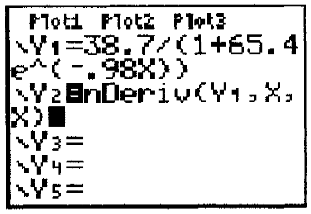

Figure 17: Derivative for regression equation

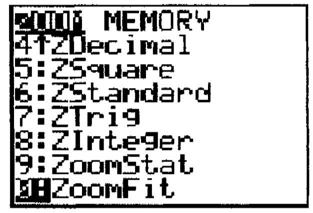

Figure 18: Window for graph of derivative

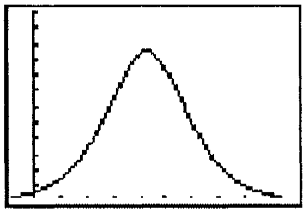

Figure 19: Graph of derivative

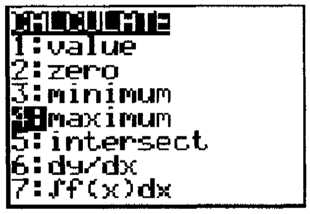

Figure 20: Find the max value of derivative

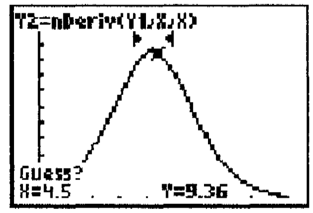

Figure 21: Left and right bounds, & guess for max

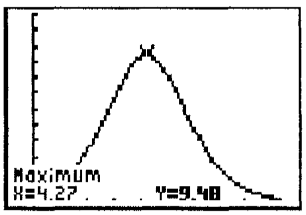

Figure 22: Results for max

It is noted here that the x-value for the max of the first derivative is the x-value of the inflection point for the logistic equation in Y1. Instructors may wish to have students find the inflection point analytically using the second derivative of the logistic equation. Also, according to the features of the logistic growth model in general, the y-value of the inflection point is given by L/2 (here, c/2 using the TI's notation). The following figures demonstrate these results; results are approximate due to the TI's numerical algorithms and round-off.

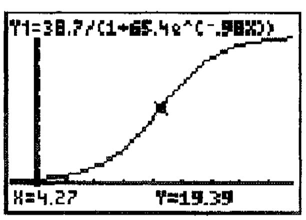

Figure 23: Inflection point

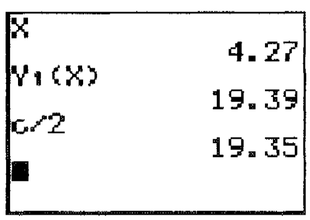

Figure 24: Y-value of inflection point and L/2 (c/2)

This tells us that the rate of U.S. DVD hardware sales was growing the fastest when about 19.35 million units were sold around the beginning of the second quarter in 2001. As stated above, the rate was about 9.48 million units per year.

3. At what rate was U.S. DVD hardware sales growing in 2003? We can answer this question in a few different ways. Analytically, students can find the first derivative of the logistic equation and then evaluate it at x=6 (coded for 2003). With the TI, we can use the numerical derivative stored in Y2 which is displayed in Figure 25. Another option is to use the differential equation associated with the logistic growth model which is dY/dX = kY(L-Y). We must note that k=b/L or with the TI's notation, k=b/c. This is shown in Figure 26. Again, results are close and approximate. Sales were growing at a rate of about 4.96 (or 4.97) million units per year in 2003.

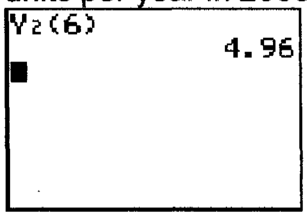

Figure 25: Numerical derivative

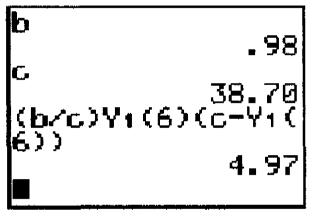

Figure 26: Differential equation

These are some of the questions that can be posed and answered for logistic growth models. Below is a list of basic questions to explore according to the logistic growth model obtained.

1. For a given input value, predict the associated output value; "near future" so extrapolate.
2. For a given input value, predict the associated output value; interpolate: how "close" or "reasonable" is the model's predicted output value to the actual output value in the data set.
3. Find an input value such that its associated output value equals/exceeds/falls below a given output value of interest.
4. Find the rate of change of output with respect to input at a particular input value of interest: find the model's derivative and evaluate it a particular input value of interest; use the differential equation associated with the model and the input value of interest.
5. Estimate the carrying capacity; horizontal asymptote or upper limit vs. "maximum value".
6. Find where the rate of change of output with respect to input is the greatest.
7. Find the differential equation whose solution is the model.
8. Given the differential equation as a model, find the particular solution of the differential equation.

Sources

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