

USING INTERNET APPLETS, GSP, AND OTHER REPRESENTATIONS TO INCREASE STUDENT UNDERSTANDING OF THE GEOMETRIC MEAN

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Introduction

The concept of geometric mean appears fairly frequently, but not consistently, in the mathematics curriculum from middle school on through high school. The concept permeates through the five content areas of number sense, algebra, measurement, geometry, and data analysis. It has the potential to be a strong connecting theme in the mathematics curriculum from middle school on through high school. However, the concept is often neglected, and for many students, it remains an obscure notion, surfacing only occasionally, without meaning, and disconnected to other important mathematics. The purpose of this article is to lift up ten characterizations of the geometric mean and to provide an interactive *Geometer's Sketchpad*® (GSP) sketch for each¹, designed to *improve understanding* – understanding of important mathematics. The sketches may be used for teacher demonstrations or by students for exploration. Ultimately, the goal is not just to increase understanding of the geometric mean, but to use the concept of geometric mean as a connecting theme and to improve understanding of important, central concepts such as multiplicative thinking, relationships in right triangles, similarity, square root, proportional reasoning, geometry of circles, and area. Finally as an extension, we conclude this article with two comprehensive GSP sketches which not only show the geometric mean of two given quantities, but also the harmonic and arithmetic means of the quantities, and we'll describe a rather striking property of the geometric mean.

Ten Characterizations of the Geometric Mean

Our purpose then is to enhance understanding of mathematics by making connections between mathematical concepts and representations of those concepts (print and electronic). For each of the following characterizations of the geometric mean, there will be

- a title of the characterization,
- a statement of the characterization,
- a brief discussion,
- comments on the proof of the statement.

Characterization #1 will be our definition and starting point.

1. **Square root of the product:** Given two numbers, the geometric mean of the numbers is the square root of the product.² By itself, this is only procedural and not very

¹ Due to space constraints, we'll only be able to show here a few of the GSP sketches which we demonstrated at ICTCM 2006 in Orlando (all sketches will be described). Furthermore, we found that GSP worked well for all the characterizations, so the Internet applets will be omitted.

² The geometric mean can be extended to a set of n numbers (the geometric mean being n^{th} root of the product), but this is beyond our focus here.

meaningful. However, we will find this calculation to be a reoccurring theme. Since the geometric mean should be *geometric*, so we proceed to the following.

2. **Squaring the rectangle:** Given a rectangle with sides a and b , the geometric mean of a and b is the length of the side of the square with the same area. A nice exploration is to give middle school students a rectangular region made up of color tiles (3 H 12, for example) and ask them to rearrange the tiles into a square. This is a nice visual representation of square root. This exploration extends nicely to *non-perfect* squares. For example, 24 tiles in a 3 H 8 rectangle can *almost* go into a square—one short of a 5 H 5 square. We'd have to *shave off* a thin slice of the outside squares and put them together to fill in the gap—making the square root of 24 (and the geometric mean of 3 and 8) slightly less than 5. *Proof:* Area of the rectangle = Area of the square; $ab = x^2$; $\sqrt{ab} = x$. The third primary characterization involves proportional reasoning.

3. **Mean proportional:** In a proportion, with a and b as “extremes” ($\frac{a}{x} = \frac{x}{b}$) the geometric mean of a and b is the value of x which solves the proportion. The geometric mean is the number that “gets used twice” in the proportion. *Proof:* cross multiply.

4. **Squares on a rectangle:** If squares are placed on two consecutive sides of a rectangle, then the area of the rectangle is the geometric mean of the areas of the squares (Figure 1). This has a *feel* of the Pythagorean Theorem, but instead of squares on legs of a right triangle, we have squares on consecutive sides of a rectangle. If the areas of the squares are r^2 and s^2 , respectively, then the area of the rectangle (geometric mean) is rs . The proof is not difficult. The next characterization involves a sequence—of course a geometric sequence.

5. **Middle number in a geometric sequence:** If a and b are the first and third numbers in a geometric sequence (a sequence formed by multiplying each term by a constant to determine the next number), then the geometric mean of a and b is the second number in the sequence. Example problems: in the following, the second and third numbers in the list are found by multiplying the previous number by a constant. Find the middle number 10, __, 90 (easy); 9, __, 16 (medium); 10, __, 20 (harder). *Proof:* Given a and b as the first and third numbers in a geometric sequence. Let x be the common ratio (“multiplier”). Then the first three numbers are a , ax , $ax^2 = b$. Solving the equation involving b , gives $x = \sqrt{b/a}$. Then the second term is $ax = \sqrt{ab}$. The next characterization is a generalization of characterization #5, and a useful thing for algebra students who are studying exponential functions to think about.

6. **Exponential function value, half way through:** Given $f(m)$ and $f(n)$ for a given exponential function f , the geometric mean of $f(m)$ and $f(n)$ is the function-value resulting from the input equal to $\frac{1}{2}$ the sum of m and n . That is, the geometric mean of the two exponential function values $f(m)$ and $f(n)$ is $f((m+n)/2)$. See Figure 2. The proof uses an arbitrary exponential function $f(x) = s \cdot r^x$, the definition from characterization #1, and properties of exponents. The remaining four characterizations are from geometry. We begin with geometric mean properties in right triangles, which are commonly found in high school geometry textbooks.

7. The altitude to the hypotenuse: Given a right triangle with an altitude drawn to its hypotenuse, the geometric mean of the lengths of the two segments created along the hypotenuse is the length of the altitude. The proof of this characterization, based on similar triangles, can be found in most high school geometry textbooks. Characterization 8 is a second geometric mean property commonly found in high school geometry textbooks. We've chosen to have two variations of characterization 8.

8a. Each leg of a right triangle: Given a right triangle, each leg of the right triangle is the geometric mean of the hypotenuse and the corresponding segment on the hypotenuse (which is the projection of the leg on to the hypotenuse). The proof of this characterization is also based on similar triangles. Characterization 8b takes advantage of the fact that a tangent segment and radius to the point of tangency form a right triangle. The tangent segment will then be the geometric mean, giving us the following.

8b. Tangent segment relative to the right triangle formed by the center of a circle, point in the exterior, and point of tangency: Given a point in the exterior of a circle, a tangent segment, and the segment drawn to the center of the circle, the length of the tangent segment is the geometric mean of the distances from the exterior point to the center of the circle and to the foot of the altitude dropped from the point of tangency to the segment drawn to the center. The third theorem about the geometric mean from high school geometry provides characterization 9.

9. Tangent segment relative to the segments on a secant from the same exterior point: Given a point in the exterior of a circle (from which there is a tangent segment and a secant segment), the length of the tangent segment is the geometric mean of the entire secant and the portion of the secant outside the circle. The proof of this characterization is also based on similar triangles (which may *not* be right triangles).

10. Length of the tangent segment of two tangent circles: Given two circles, with diameters a and b , which are externally tangent to each other and both tangent to a line, the length of the segment between the two points of tangency (on the tangent line) is the geometric mean of the diameters. This is an especially nice diagram because the length of the segment between the centers is the arithmetic mean (by virtue of the fact that radii are half the diameters) and the segment joining the points of tangency is the geometric mean, and can be seen to be smaller than the arithmetic mean. The proof utilizes the Pythagorean Theorem and a bit of algebra.

Two GSP Sketches Connecting the Harmonic, Geometric, and Arithmetic Means

As an extension and a combination of some of the characterizations, we now share two more *GSP* sketches, each showing the relationship between the harmonic (the harmonic mean of a and b is $2ab/(a+b)$), geometric, and arithmetic means (for a and b , the arithmetic mean is $(a+b)/2$). These sketches all share the following characteristics.

- In the sketch are the two initial quantities, a and b , and the three means,
- Harmonic mean # geometric mean # arithmetic mean, *and*
- The geometric mean is not only the geometric mean of a and b , but it is *also* the geometric mean of the harmonic and arithmetic means themselves. The latter is true because the three means will be in a right triangle as in characterization 8a. In particular, the arithmetic mean is the hypotenuse, the

geometric mean is a leg, and the harmonic mean is the corresponding portion on the hypotenuse.

All Three Means I - Using Characterizations 7 & 8. Given a right triangle with an altitude drawn to its hypotenuse, the geometric mean (GM) of the lengths of the two segments created along the hypotenuse is the length of the altitude. Furthermore, the arithmetic mean (AM) is the radius of the circle formed with the hypotenuse as the diameter (label the center A), and the harmonic mean (HM) is the corresponding segment on the arithmetic mean (which is the projection of the geometric mean on to the arithmetic mean).

All Three Means II - Using Characterizations 8 & 10. Given two circles, with diameters a and b , which are externally tangent to each other and both tangent to a line, the length of the segment between the two points of tangency (on the tangent line) is the geometric mean of the diameters. Furthermore, the distance between the centers is the arithmetic mean. The harmonic mean is the corresponding segment on the arithmetic mean (which is the projection of the geometric mean on to the arithmetic mean).

Conclusion

The concept of geometric mean, in conjunction with dynamic GSP sketches, provides opportunities to show the interplay between algebra, arithmetic operations, and geometry, thereby enhancing understanding of a number of key mathematical concepts.

Reference

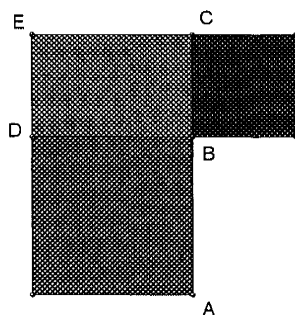
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Squares on a Rectangle

Hide Explanation
Hide Directions
Hide Geometric Mean

If squares are placed on two consecutive sides of a rectangle, then the area of the rectangle is the geometric mean of the areas of the squares.

Directions: You may manipulate the sketch by moving points A, B, and C. (To reposition the sketch click on one of the three areas and move the sketch to the desired position.)



Area of Blue Square = 4.99 cm^2

Area of RED Square = 11.83 cm^2

The Area of the Green Rectangle = 7.68 cm^2

$$\sqrt{(\text{Red Area}) \cdot (\text{Blue Area})} = 7.68 \text{ cm}^2$$

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Figure 1: Squares on a Rectangle

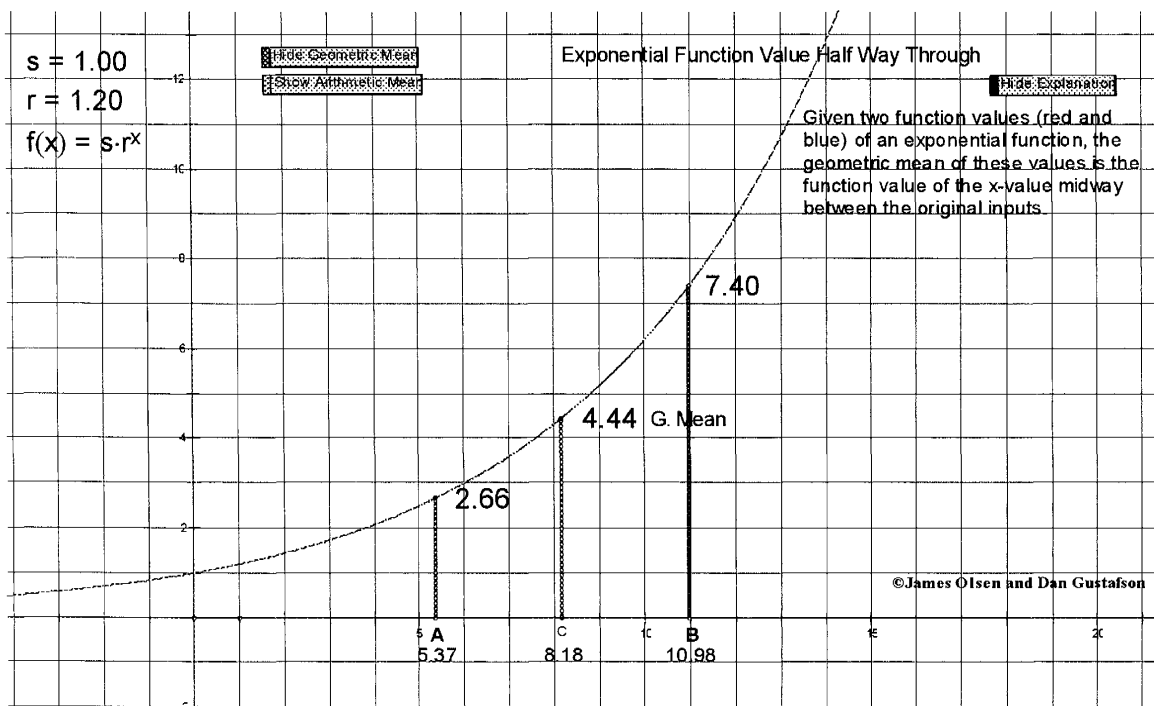


Figure 2: Exponential Function Value, Half Way Through

All Three Means II - Using Characterizations 8 & 10

$a = 4.29$

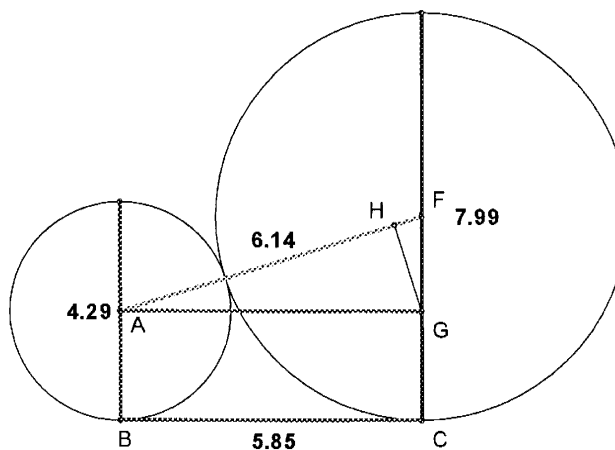
$b = 7.99$

H.M. = 5.58

G.M. = 5.85

A.M. = 6.14

Hide Explanation
 Show Directions
 Hide Lengths
 Show Harmonic Mean



Given two circles, with diameters a and b , which are externally tangent to each other and both tangent to a line, the length of the segment between the two points of tangency (on the tangent line) is the **geometric mean** of the diameters.

Furthermore, the distance between the centers is the **arithmetic mean**. The **harmonic mean** is the corresponding segment on the arithmetic mean (which is the projection of the geometric mean on to the arithmetic mean).

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Figure 3: All Three Means II – Using Characterizations 8 & 10