USING MAPLE TO DEVELOP STUDENTS' INTUITION IN SOLVING DISCRETE MATHEMATICS PROBLEMS

Kate McGivney and Doug Ensley
Shippensburg University
Mathematics Department
1871 Old Main Drive
Shippensburg, PA 17257
kgmcgi@ship.edu, deensl@ship.edu

Introduction. In this paper we describe several Maple lab activities appropriate for the undergraduate discrete mathematics course. Recommendations from the Mathematical Association of America (MAA) and the National Council of Teachers in Mathematics (NCTM) suggest that students should gain experience with computer algebra systems early in their undergraduate career. Specifically, Recommendation C.2 of the MAA's CUPM document states that mathematics departments should design major courses that "...develop skill with a variety of technological tools: Departments should ensure that majors have experiences with a variety of technological tools, such as computer algebra systems, visualization software, statistical packages, and computer programming languages"[1]. This sentiment is echoed by those responsible for the training of future teachers. In *The Mathematical Education of Teachers* [2], it is recommended that future mathematics teachers gain experience using a computer algebra system throughout their undergraduate career. In light of these recommendations, we have written a series of lab activities to be used in the freshman-level discrete mathematics course. The topics that we focus on are: sequences, recursion, induction, sets, counting, and probability.

Sequences. We begin with an activity that invites students to investigate patterns. The students are given Maple script that produces the first ten terms of a recursive sequence. Next they run the Maple script and see if they can guess the closed formula for the given sequence. The purpose is two-fold: the students practice working with both the closed and recursive forms of a sequence as well as gain experience in reading and writing loops.

Consider the sequence $a_n = a_{n-1} + 2n + 1$ for $n > 1$ and with $a_1 = 1$. Using the following Maple script, we can quickly see the first 10 terms of this sequence.

```
a:=array(1...10):
a[1]:=1:
print(1,a[1]);
for i from 2 to 10 do
  a[i]:=a[i-1]+(2*i+1):
  print(i,a[i])
od:
```

What do you think is the closed form for this sequence? Check your conjecture by modifying the above script so that the closed form replaces the recursive form.

As a follow-up activity, we provide the closed form for a sequence and ask that the students make a conjecture about the recursive form.

Sets. As a first introduction to sets and set operations, we use the following activity.

Given the universal set $U = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}$, use the following Maple commands to generate subsets *A*, *B*, and *C* of size 3, 5, and 7, respectively.

```
with(combinat, randcomb);
U:={0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15};
A:=randcomb(U,3);
B:=randcomb(U,5);
C:=randcomb(U,7);
```

Use the randomly generated sets to determine the following: $A \cup B$, $A \cap B$, $(A \cup C) \cap B$, A', $A \cap (B \cap C)$, $A - B$, $A' \cap B$, $\mathbf{P}(A)$, $A \times B$.

Next we'll see how we can use Maple to check our answers. To find the union of sets *A* and *B*, simply type **A union B;** The intersection command is **A intersect B;** To find the set $A - B$, type **A minus B**; Maple does not have a built in "complement" command. How can you use one of the above commands to find the complement? The power set of a set *A* can be found by typing the following commands:

```
with(combinat, powerset);
powerset(A);
```

Finally, the Cartesian product of sets *A* and *B* is found using the following more complex Maple command:

```
'union'(op(map(y->map(x->[x,y],A),B)));
```

Counting. In the counting unit, the students discover the relationship between the number of subsets and the number of permutations that can be formed from a set of *n* distinct objects. The purpose is to let the students view sets, count subsets, determine the equivalence classes so that they can discover the relationship between combinations and permutations. There are also extension activities that allow the students to use Maple to investigate topics such as recursive counting, the binomial theorem and Pascal's triangle.

In the first lab we develop a formula that relates the number of subsets and the number of permutations that can be formed from *n* distinct objects. Recall that the difference between subsets and permutations is that in the first case we don't care about the order of the entries and in the second case we do. Helpful counting discrete math Maple commands are bundled together. To access these commands, type: **with(combinat);** The Maple command **permute(n,r);** generates the permutations of size *r* from a set of *n* distinct elements. Recall that $P(n, r)$ is used to represent the number of permutations of size *r* from the set of n objects. Execute **permute(n,r);** for the following choices of *n* and *r*.

n	3	3	3	3
r	0	1	2	3

What is $P(n,r)$ in each case?

Next, we'll compute the number of subsets of size r from a set of n distinct elements. Since now we do not care about the order of the entries, [1,2] and [2,1] are considered "equivalent." Count the number of subsets that arise in each of the above cases.

The Maple command **choose(n,r);** constructs the subsets of size r that can be formed from n objects. The *number* of subsets of size r that can be formed from n objects is denoted by C(n, r). Calculate C(3, r) for $r = 0, 1, 2, 3$.

n	3	3	3	3
r	0	1	2	3
P(n, r)				
C(n, r)				

Now let's look at a slightly larger set, one that contains 4 distinct entries. Use the **permute(n,r)** and the **choose(n,r)** commands to find all permutations (subsets) of size r. Determine the number of permutations (subsets) for each choice of $r = 0,1,2,3,4$. Can you use your results to come up with a formula that relates P(n, r) and C(n, r)? Explain why this formula makes sense.

The next problem links the concepts of recursion and counting. The main question is: "Why is P(n, r) = $n \cdot$ P(n – 1, r – 1)?" Consider another question, "How many permutations of length r use entries from $\{1,2,\ldots,n\}$?" We know the answer is P(n, r), but let's see another way to get this value. To answer this, we can use a two-step decision process: (1) Choose a first entry from $\{1,2,\ldots,n\}$; (2) Fill in the remaining $r – 1$ entries with a permutation of length $r – 1$ sing elements from the other $n – 1$ elements of the set. Notice that this relationship does not help if $r = 0$, so it's necessary to specify that P(n, 0) = 1 for this recursive formula to make sense. The following Maple program implements this idea:

```
P := proc(n,r)
option remember:
if (r=0) then RETURN(1)
else RETURN(n*P(n-1,r-1))
fi
end;
```

Find P(6, 2) two ways: (1) using the above procedure and (2) using the Maple command **permute(6,2)**. How do the answers compare? Next modify the above procedure to explore if the following is true:

$$P(n, k) = P(n - 1, k) + P(n - 1, k - 1).$$

Probability. To help students develop intuition on thinking of probability as a "long-term relative frequency" and expected value as a "long-term average", we use Maple to simulate random events so that students can use data to form conjectures. Often students are presented with the topic of probability as an extension to counting problems and therefore bypass the simulation step and go directly to calculating theoretical probabilities. By omitting the simulation step, students often do not develop an intuitive feel as to the probability of a particular random event. With this in mind, we have developed a series of activities that invites the students to simulate random events using Maple, devise empirical probability estimates, make conjectures about the theoretical probability, and then use their knowledge of combinatorics to verify the conjecture. One such activity follows:

Question. Given that there are 30 of us in the room, what is the probability that at least two of us share the same birthday?

We will use the command **randlist(n,k);** which returns a random set of k values from a set of n objects. What are n and k in this situation? Simulate the birthday problem one time. Was there at least one match?

To get an estimate of the likelihood that there will be at least one match we need to perform many simulations. Each group will now simulate this experiment 20 times, keeping track of whether there was at least one match each time. After everyone is done, we'll pool our data to determine the empirical probability that there's at least one match. To more easily determine whether or not there is a match, we'll sort the data each time. To perform the 20 simulations, type:

sort(randlist(365,k)); (where k is the number of students present)

Trial	1	2	3	4	5	6	7	8	9	10	11	12	13
Match?													
Trial	14	15	16	17	18	19	20						
Match?													

Based on the pooled data, what is the empirical probability that there'll be at least one match? Finally, find the theoretical probability that there will be at least one match. (*Hint*: Find the probability of the complementary event, that is, find the probability that given 30 people, there are no matches.) How close was your empirical estimate to the true probability?

After students have gained experience with using simulation to estimate theoretical probabilities, we use simulation to get at the heart of expected value. If students are solely presented with the definition of expected value for a discrete random variable, that is, the sum of each outcome times its corresponding probability, they often do not develop a clear picture as to the meaning of this number. If instead this concept is presented as a "long term average" and the students are able to simulate the event for many trials and compute an average, they develop a deeper understanding of expected value. A sample activity is included below.

Question. Suppose five cards including an Ace are drawn from a standard deck. (If the five cards do not include an Ace, they are replaced and the deck is reshuffled.) What is the expected number of aces among the five cards?

We'll approach this problem using simulated data first. Begin by using Maple to randomly select 5 cards from a fair deck of 52, where one card is an ace. The following script generates 20 5-card hands from a fair deck of 52.

```
with(combinat, randcomb);
carddeck:={Kc,Kh,Ks,Kd,Qc,Qh,Qs,Qd,Jc,Jh,Js,Jd,seq(i.c,
i=2..10),seq(i.h,i=2..10),seq(i.s,i=2..10), seq(i.d,i=2..10)};

aces:={Ac,Ah,As,Ad};
for i from 1 to 20 do
sort(randcomb(aces,1),randcomb(carddeck,4),lexorder);
end do;
```

What is the average number of aces based on your simulation data? Pool your data with four friends. What is the average number of aces based on your groups' data? Suppose we were to simulate this experiment infinitely many times. What is your guess as to the average number of aces in a 5-card hand? Now solve this problem using *theoretical probabilities*.

Conclusion. For more Maple lab activities on sequences, recursion, induction, sets, counting, and probability see
http://www.ship.edu/~deensl/DiscreteMath/ictcm03/index.html

References

[1] Pollatsek, Harriet, et.al., *Undergraduate Programs and Courses in the Mathematical Sciences: A CUPM Curriculum Guide*, Draft 3.2, January 2003.

[2] *The Mathematical Education of Teachers*, Conference Board of the Math Sciences Issues in Mathematics Education – Volume 11, AMS, 2001.

[3] D. Ensley, J. Crawley, *Introduction to Discrete Mathematics: Mathematical Reasoning with Puzzles, Patterns, and Games*, to appear from John Wiley and Sons, Spring 2005.