THE CUBIC AND QUARTIC EQUATIONS IN INTERMEDIATE ALGEBRA COURSES

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I. Introduction

A. Historical background

The solution of the cubic and quartic equations is important in the history of mathematics for several reasons. First, it was the first major advance by modern man since the time of the ancient Greeks. It was the first mathematical formula unknown to the ancients. Second, it forced mathematicians to take both complex numbers and negative numbers seriously. And, more importantly, it led to the study of the theory of equations, culminating in the nineteenth century in the proof of the insolvability of the quintic.

There are a number of key figures in the triumph of the cubic and quartic formulas. Omar Khayyam (1048-1123) used intersections of conics to give geometric constructions of roots of cubics. Leonardo of Pisa (Fibonacci) (c1180-1245) had an approximation formula for certain forms of the cubic. About 1515, Scipione del Ferro (d1526) discovered a method for finding the roots of several forms of the cubic and shared these secrets with some of his students. Antonio Ma Flor, one of these students, challenged mathematicians, in 1535, to a problem-solving contest involving the cubic. Niccolo Fontana (Tartaglia, 'the stammerer') (c1500-1557) answered the call and found the general solution. Giralamo Cardano (1501-1576), published the secret formula in his treatise *Ars Magna* (The Great Art, or The Rules of Algebra), in 1535, with due credit to Tartaglia.

The quartic formula was discovered by Lodovici Ferrari (1522-1565) in 1540. A similar formula was found by Rene Descartes (1596-1650) around the same time.

For hundreds of years, mathematicians sought some generalization of the classical formulas that would give the roots of any polynomial. Finally, P. Ruffini (1765-1822), in 1799, and Niels Henrik Abel (1802-1829), in 1824, proved that no such formula exists for the general quintic. Evariste Galois (1811-1832) was able to determine precisely those polynomials whose roots can be found (and, in so doing, founded the theory of groups).

B. The quadratic equation

The solution of the quadratic equation: $ax^2+bx+c=0$ is familiar to every student of intermediate algebra:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

with the discriminant Δ = {+, two real roots; -, two complex roots; 0, one double root} and with the properties: $x_1+x_2=-\frac{b}{a}$; $x_1 \bullet x_2 = \frac{c}{a}$.

II. Solution to the Cubic

The solution to the cubic: $ax^3+bx^2+cx+d=0$ will be patterned after the quadratic formula. Assume: $a=1$. The discriminant Δ $=(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2$ = {+, three real roots; -, two complex roots; 0, multiple roots}.

<u>Step 1.</u> Let $x=y-\frac{b}{3}$. $\Rightarrow$ $y^3+py+q=0$ (reduced equation). Then

$$p=c-\frac{b^3}{3} \qquad q=d-\frac{bc}{3}+\frac{2b^3}{27}.$$

<u>Step 2.</u> Let $y=z-\frac{p}{3z} \Rightarrow$ $(z^3)^2+q(z^3)-\left(\frac{p}{3}\right)^2=0$ $\Rightarrow$ $z^3=-\frac{q}{2}\pm\sqrt{\frac{p^2}{9}+\frac{q^2}{4}}$

Choose: $z_1z_2=-\frac{p}{3}$, then: $y_1=z_1+z_2$ $\quad y_2=\omega z_1+\omega^2 z_2$ $\quad y_3=\omega^2 z_1+\omega z_2$

$\omega=\frac{1}{2}+i\frac{\sqrt{3}}{2}$, complex cube root of unity, with identities: $\omega^3=1$; $1+\omega+\omega^2=0$.

$\Rightarrow y_1+y_2+y_3=0$; $y_1y_2+y_2y_3+y_3y_1=p$; $y_1y_2y_3=-q$ $\Rightarrow$ $\Delta=-4p^3-27q^2$.

<u>Case 1.</u> $\Delta=0$

$$z_1=z_2=\sqrt[3]{-\frac{q}{2}}$$

$y_1=z_1+z_2=\sqrt[3]{-4q}$; $\qquad y_2=\omega z_1+\omega^2 z_2=(-1)z_1=\sqrt[3]{\frac{q}{2}}$; $\qquad y_3=y_2$.

Case 2. $\Delta < 0$.
$$z_1, z_2 = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\left(\frac{p}{3}\right)^2 + \left(\frac{q}{2}\right)^2}}$$

$$y_1 = z_1 + z_2\,; \qquad y_2, y_3 = \frac{1}{2}(z_1+z_2) \pm \frac{i}{2}(z_1 - z_2)\sqrt{3}$$

Case 3. $\Delta > 0$. $\Rightarrow p < 0.$ Reduced equation: $y^3 + py + q = 0 \Rightarrow$

Let $y = 2\sqrt{-\frac{p}{3}}\,z,$ and using the identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta,$

$$\Rightarrow \quad 4z^3 - 3z = k \qquad z = \cos\theta; \qquad k = -\left(\frac{q}{2}\right)\left(-\frac{3}{p}\right)^{3/2}.$$

$$\therefore y_n = 2\sqrt{-\frac{p}{3}}\cos\left(\theta + \frac{2n\pi}{3}\right), \quad n = 0,\ 1,\ 2.$$

III. Solution to the Quartic $x^4 + bx^3 + cx^2 + dx + e = 0.$

A. Ferrari

Let $x = y - \frac{b}{4}$ $\Rightarrow y^4 + py^2 + qy + r = 0,$ or $y = -py^2 - qy - r.$

Add: $y^2 z + \frac{z^2}{4}$ to both sides:

(1*) $$\left(y^2 + \frac{z}{2}\right)^2 = (z-p)y^2 - qy + \left(\frac{z^2}{4} - r\right) = (my+k)^2; \quad \text{m, k to be determined.}$$

$$\Rightarrow y^2 + \frac{z}{2} = \pm(my+k).$$

In (*1), RHS is quadratic in y^2. It is a perfect square if discriminant is zero, ie:

$$q^2 - 4(z-p)\left(\frac{z^2}{4} - r\right) = 0, \quad \text{or} \quad (*2) \quad z^3 - pz^2 - 4rz + \left(4pr - q^2\right) = 0,$$

called the *resolvent cubic.*

Ex. 1. $y^4 + 3y^2 - 2y + 3 = 0.$ (*2) becomes:

$z^3 - 3z^2 - 12z + 32 = 0,$ which has root: z = 4. Thus: $\left(y^2+2\right) = y^2 + 2y + 1 = (y+1)^2$

$$y^2 + 2 = +(y+1) \quad \Rightarrow \quad y = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$y^2 + 2 = -(y+1) \quad \Rightarrow \quad y = \frac{1}{2} \pm i\frac{\sqrt{11}}{2}.$$

B. Descartes: reduced quartic

$$y^4 + py^2 + qy + r = 0.$$

Factor: (*3) $\left(y^2+ky+m\right)\left(y^2-ky+n\right) = y^4 + \left(m+n-k^2\right)y^2 + \left(kn-km\right)y + mn$

$\Rightarrow m+n-k^2 = p \; m+n = p+k^2 \qquad k(n-m) = q \qquad mn = r$

$\Rightarrow \qquad 2n = p+k^2+q/k \qquad\qquad 2m = p+k^2-q/k$

$4r = 2n2m = \left(p+k^2+\frac{q}{k}\right)\left(p+k^2-\frac{q}{k}\right)$, or: $\left(k^2\right)^3 + 2p\left(k^2\right)^2 + \left(p^2-4r\right)k^2 - q^2 = 0.$ (*4)

Any root of (*4) gives a factorization (*3).

Ex. 2. $y^4-3y^2+6y-2=0 \qquad \Rightarrow \left(k^2\right)^3 - 6\left(k^2\right)^2 + 17k^2 - 36 = 0$ with root $k^2 = 4$. Thus:

$y^4-3y^2+6y-2 = \left(y^2+2y-1\right)\left(y^2+2y+2\right)$

$\therefore y_1 = -1+\sqrt{2}; \qquad y_2 = -1-\sqrt{2}; \qquad y_3 = 1+i; \qquad y_4 = 1-i.$

IV. Examples

Ex. 1. $x^3+7x^2+11x+5=0$

$b=7, c=11, d=5 \qquad \Rightarrow \qquad p = c-\frac{b^2}{3} = \frac{16}{3}; \qquad q = d-\frac{bc}{3}+\frac{2b^3}{27} = \frac{128}{27}$

$\Rightarrow \qquad \Delta = -4p^3-27q^2 = 0 \qquad \Rightarrow$ multiple roots:

$y_1 = \sqrt[3]{-4q} = -\frac{q}{3} \qquad y_2 = y_3 = \sqrt[3]{\frac{q}{2}} = \frac{4}{3} \qquad \therefore x = y-\frac{b}{3} \qquad x_1 = -5, \qquad x_2 = x_3 = -1$

Ex. 2. $x^3-6x-9=0$

$x = y \qquad \Rightarrow p=-6; q=-9 \qquad \Rightarrow \qquad \Delta = -1323 \qquad \Rightarrow$ two complex roots

$\therefore x_1 = z_1+z_2 = 3; \qquad x_2, x_3 = \frac{1}{2}\left(z_1+z_2\right) \pm \frac{i}{2}\left(z_1-z_2\right)\sqrt{3} = \frac{1}{2}\left(-3 \pm i\sqrt{3}\right)$

Ex. 3. $2x^3-5x^2-x+6=0$

$\Rightarrow x^3-\frac{5}{2}x^2-\frac{1}{2}x+3=0 \;\Rightarrow\; b=-\frac{5}{2}, c=-\frac{1}{2}, d=3 \;\Rightarrow\; p=-\frac{31}{12}, q=\frac{77}{54}$

$\Rightarrow \qquad \Delta = \frac{225}{16} \qquad \Rightarrow$ three real roots

$$\theta=\frac{1}{3}\cos^{-1}\left[\left(-\frac{q}{2}\right)\left(-\frac{3}{p}\right)^{3/2}\right]=0.89102 \qquad y_n=2\sqrt{-\frac{p}{3}}\cos\left(\theta+\frac{2n\pi}{3}\right)$$

$$y_1=\frac{7}{6};\quad y_2=-\frac{11}{6};\quad y_3=\frac{2}{3}; \qquad \therefore x_1=y_1-\frac{b}{3}=2;\quad x_2=-1;\quad x_3=\frac{3}{2}.$$

Ex. 4. (*Curio):* $x^3+x^2-2=0$

$$\Rightarrow b=1, c=0, d=-2 \quad \Rightarrow \quad p=c-\frac{b^2}{3}=-\frac{1}{3};\ q=d-\frac{bc}{3}+\frac{2b^3}{27}=-\frac{52}{27}$$

$$\Rightarrow \quad \Delta=-4p^3-27q^2=-\frac{212}{27}$$

$$z_1,z_2=\sqrt[3]{-\frac{q}{2}\pm\sqrt{\frac{p^3}{27}+\frac{q^2}{27}}}=\frac{1}{3}\sqrt[3]{26\pm15\sqrt{3}};\ y_1=z_1+z_2=\frac{1}{3}\left(\sqrt[3]{26\pm15\sqrt{3}}+\sqrt[3]{26-15\sqrt{3}}\right);$$

$$y_2,y_3=-\frac{1}{2}(z_1+z_2)\pm\frac{i\sqrt{3}}{2}(z_1-z_2)=-\frac{1}{6}\left(\sqrt[3]{26+15\sqrt{3}}+\sqrt[3]{26-15\sqrt{3}}\right)\pm\frac{i\sqrt{3}}{6}\left(\sqrt[3]{26+15\sqrt{3}}-\sqrt[3]{26-15\sqrt{3}}\right)$$

$$\therefore x=y-\frac{1}{3};\quad x_1=\frac{1}{3}\left(\sqrt[3]{26+15\sqrt{3}}+\sqrt[3]{26-15\sqrt{3}}-1\right)$$

$$x_2,x_3=\frac{1}{6}\left(\sqrt[3]{26+15\sqrt{3}}+\sqrt[3]{26-15\sqrt{3}}+2\right)\pm\frac{i\sqrt{3}}{6}\left(\sqrt[3]{26+15\sqrt{3}}-\sqrt[3]{26-15\sqrt{3}}\right)$$

But the original equation has the root: x = 1.

$$\Rightarrow\left(\sqrt[3]{26+15\sqrt{3}}+\sqrt[3]{26-15\sqrt{3}}\right)=4 \quad \text{and} \quad \left(\sqrt[3]{26+15\sqrt{3}}-\sqrt[3]{26-15\sqrt{3}}\right)=2\sqrt{3}.$$

V. Conclusion

Any polynomial equation of degree ≤ 4, with real coefficients, is solvable by radicals or complex numbers. When the solution is impractical, we use other means, like the Rational Root Theorem, Descartes' Rules of Signs, or approximation methods, like Newton's approximation method.

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