MODELING AND APPLICATIONS IN NUMERICAL ANALYSIS

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Modeling and applications have become a central theme for all courses taught by these authors. With eighteen years experience of using projects in numerical analysis, Fox and West have started a manuscript for a book of projects that can be used in numerical analysis courses or integrated into college algebra and/or calculus-based courses. Over approximately the past ten years, both computer and graphing calculator technology have changed the landscape of student learning in many areas, especially in numerical analysis. Our presentation at the ICTCM was based on using the TI-83 Plus as a "lowest common denominator" of technology. We felt like this was technology accessible to all and would be most useful to teachers of topics related to numerical computing.

Background. We served together at the Department of Mathematical Sciences at the United States Military Academy at West Point, NY, from 1990 through 1998. This was a period of great change in mathematics at West Point. The emphasis of all the mathematics curricula went from traditional pencil and paper skills to a technology-supported modeling approach. During this period West was the director of a numerical analysis two-course-sequence that had been developed back in the early 1980's, and Fox was the developer and writer for the modeling thread for our mathematics majors. The projects that we presented at our workshop have been used in many venues over the passed twenty years. We continue to see new uses for these projects while teaching at Francis Marion University.

Calculator Interface. With the current technology there are many approaches that can be taken toward the representation and understanding of a numerical solution to a problem. Generally, the numerical representation of a solution to a problem is a table of solution values, which may be exact or approximate. The four approaches that current technology enhances are: (1) the graphical or geometric approach, (2) the recursive or iterative approach, (3) the discrete dynamical system (or sequence) approach, and (4) the programming approach. The graphical approach aids in student understanding of approximation. The TRACE and TABLE modes of the graphical calculator interface the general picture of the graph and provide the user with the specific numerical values. The recursive and iterative approach show how a spreadsheet works to produce a numerical solution. A discrete graph of these values gives a more general picture of an underlying function. Most algorithms in numerical analysis can be written as a discrete dynamical system or as a system of difference equations. We have found that this discrete approach to modeling is intuitive to students and with minimal introductory work is accessible to most students. Finally, a more traditional approach to numerical solutions is to write programs for the algorithms as they are developed. This is a less accessible aspect of the calculator, but very valuable to students for understanding computer logic and the student ownership of developing their own programs. We presented all these approaches at our workshop and encourage teachers to choose the approach that best suits their course objectives.

Projects. As we said above, many of the projects presented here were developed over the last twenty years at West Point. We present these projects in the context of their disciplines other than mathematics, so that they are relevant to students or to their future. Thus, we call these projects interdisciplinary. We often use these projects in a summative manner, after initial learning takes place, to cement student understanding. However, we have used these projects in a formative manner, to enhance the learning of something new. Finally, we have occasionally used these projects as part of the course, where students do their own research and learning through doing the projects.

In the workshop, we couldn't cover all the topics of numerical analysis, so we chose those that would highlight the TI-83 Plus and modeling. The topics we chose were solutions to single-variable equations, solutions to initial value problems, numerical methods in optimization, approximations, a little linear algebra, and systems of difference equations.

Root Finding

When a machine is t years old, it earns revenue at a rate of e^{-t} dollars per year. After t years of use the machine can be sold as scrap for $\frac{1}{t+1}$ dollars. How long should the company own the machine in order to maximize total revenue?

$$R(x) = \int_0^T e^{-t}\,dt + \frac{1}{t+1}$$

To maximize revenue, we take the derivative and set it equal to zero.
$R'(x) = e^{-t} - (t+1)^{-2} = 0$

Here, we have a transcendental function that does not have a closed form solution. Let's try numerical methods using the TI-83 Plus.

- Newton's Method
- Internal calculator routines under, **Plot**, **Calc** zeroes (for example)

We find that $t=2.512862417$ is the value where $R'(2.512862417)=0$.

Initial Value Problems

Given the differential equation model for the spread of a communicable disease or a population model as:

$$\frac{dN}{dt} = .25N(10 - N),\ N(0)=2$$

Let's completely analyze its behavior.

(a) Since this is an autonomous ordinary differential equation (ODE) perform a complete graphical analysis:
   (1) Plot *dN/dt* versus *N*. Find and label all rest points.
   (2) Find the value where the rate of change of the disease is the fastest. Why did you provide this value?
   (3) Plot *N* versus *t* from the following initial conditions: $N(0) = 2$, $N(0) = 7$, and $N(0)=14$.
   (4) Describe the *stability* of each rest point.

(b) Solve this ODE using separable variables. (Hint: You might need partial fraction decomposition as well) Ensure you find the value of arbitrary constant *c* using the initial condition $N(0)=2$.

(c) Using your analytical solution in (b), plot *N* versus *t*.

(d) Compare your actual plot with the graphical solution in (a.3).

(e) From your graph in (c), estimate the solution value of $N(.5)$ and $N(5)$.

(f) Find the actual values for these two solutions $N(.5)$ & $N(5)$ using your analytical solution from part (b).

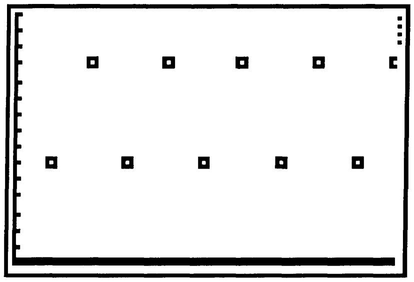
(g) Compute the time, *t*, when N is changing the fastest using the initial condition N(0)=2.

(h) Use Euler's Method with $h = 1$, $h=0.5$ and then $h = 0.1$ to approximate the solution to the ODE for $N(.5)$ & $N(5)$. Find the relative error or absolute differences.

(i) Plot Euler's approximations for h=0.1, h=0.5, and h=1.0 and compare these graphs to your graphical analysis in part (a) and your plot in part (d). Briefly discuss these plots -- you may compare and contrast these plots.

(j) Repeat parts (h) and (i) using Runge-Kutta Methods.

This exercise shows the importance of step-size in obtaining "good" numerical solutions. As a result, when the step-size is 1, we obtain a poor approximate solution as in Figure 1. When we change the step-size to 0.5, we obtain a much better approximate graph to the solution as in Figure 2. This solution follows the qualitative graphical solution we initially would find.



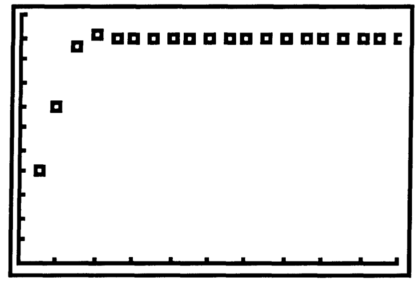


Figure 1. Poor Solution

Figure 2. Better Approximate Solution

Optimization

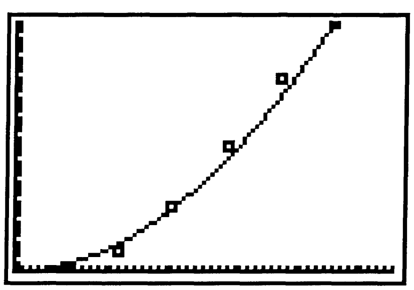
Consider fitting the model, $y=a\,x^2$ to the data provided using the sum of the absolute deviations as our criterion.

X	7	14	21	28	35	42
Y	8	41	133	250	388	497

Minimize = |8-49a|+|41-196a|+|133-441a|+|250-874a|+|388-1225a|+|497-1764a|

The method we are using is the Method of Golden Section that allows us to find a small interval solution to a unimodal function.

The interval (0.28549,0.28635) is found. The midpoint is 0.28592.
Here is the plot of the original data and the function found by minimizing the sum of the absolute deviations.



Systems of DEs (differential equations or difference equations). The two projects presented below can be modeled as systems of differential equations or as systems of difference equations (previously called discrete dynamical systems). For the workshop we chose to model them as difference equations to show the capabilities of the TI-83 Plus graphing calculator.

A Fishy Problem. Given a number of assumptions, growth rates of trout and bass in isolation, and factors of how interactions affect the populations of trout and bass in a pond, can these two species of fish coexist in the pond?

- Growth rates for trout and bass in isolation are 15% and 10%, respectively.
- Decrease in trout due to competition is $.015 \times$ no. of trout $\times$ no. of bass.
- Decrease in bass due to competition is $.011 \times$ no. of trout $\times$ no. of bass.
- Initial quantities of trout and bass are 15,000 each.

(a) Model the population of the trout and bass as a system of difference equations.

(b) Graph the populations of trout *vs.* time and bass *vs.* time for at least fifty years or seasons. Which population dies off?

(c) Graph the population of trout *vs.* the population of bass. Again, which population dies out? Does it appear that the two populations are approaching an equilibrium point?

(d) If it exists, find this equilibrium point where both populations could coexist.

(e) Now try some different initial populations and graph the phase portraits (trout *vs.* bass). For instance, let the initial trout population equal 5,000, 8,000, 15,000 and 20,000 for a bass population of 5,000, 8,000, 15,000, and 20,000. That's sixteen graphs. What general trends do you see? How would these trends inform someone trying to manage the trout and bass in this pond?

Summary. The authors, as well as their students, enjoy the use of mathematical modeling to add "realism" to the mathematics being discussed. These examples listed here are just the "tip of the iceberg" of projects that we have used in a myriad of mathematics courses. The authors have developed TI-83 Plus programs, EXCEL spreadsheets, and MAPLE programs as well as examples for the following: Root Finding (Bisection and Newton's Method), Initial Value Problems (Euler and Runge-Kutta Methods), Approximation (Cubic Splines and Divided Differences), and Discrete Dynamical Systems. Contact us to receive copies or information about any programs or projects (e-mail: rwest@fmarion.edu or wfox@fmarion.edu).

References.

Burden, R. L. and Faires, J. D. (1997). Numerical Analysis, 6th Ed. Pacific Grove, CA: Brooks/Cole Publishing Company.

Ellis,W., Bauldry,W., Fiedler, J., Giordano, F., Judson, P., Lodi, E., Vitray, R., and West, R. (1999). Calculus: Mathematics and Modeling, Updated Preliminary Edition. Reading, MA: Addison-Wesley.

Fox, W. P., Giordano, F., and Weir, M. (2002). A First Course in Mathematical Modeling, 3rd Ed. Pacific Grove, CA: Brooks/Cole Publishing Company.