ELEMENTARY MATHEMATICAL MODELS FOR GUITAR FRET PLACEMENT

Thomas E. Rutenkroger
Valencia Community College
701 North Econlockhatchee Trail
Orlando Florida
trutenkroger@valenciacc.edu

A Classical Exponential Model for Guitar Fret Placement

A luthier is a woodcraftsmen who specializes in guitar building. Acoustical guitar construction has been evolving over centuries. Until the development of modern analysis of string vibration a commonly used method for guitar construction, specifically the placement of frets was the Rule of Eighteen. The Rule of Eighteen is an exponential model that determines the proper placement of a fret on the fingerboard of the guitar. The relative position of each fret determines the frequency of vibration of the string. When the string length is shortened, the frequency of vibration increases, thus creating a higher pitch. Now define the following subscripted variables:

$\mathbf{L_0}$ is the open vibrating string length which generates the fundamental frequency for each of the six strings,

$\mathbf{L_1}$ is the vibrating string length when the string is held down at the first fret, thus shortening the string length and increasing the pitch (i.e. a higher note on the given scale)

$\mathbf{L_n = L(n)}$ is the vibrating string length when the string is held down at the nth fret, thus shortening the string length and giving a pitch corresponding to the nth fret.

The Rule of Eighteen states that the first fret $\mathbf{L_1}$ should be placed at a distance that is $\frac{17}{18}$ th of the total string length $\mathbf{L_0}$ relative to the saddle. The second fret should be placed at a distance which is $\frac{17}{18}$ th of the distance from the first fret $\mathbf{L_1}$ to the saddle. The third fret should be placed at a distance which is $\frac{17}{18}$ th of the distance from the second fret $\mathbf{L_2}$ to the saddle, and so forth.

For example, if the total string length is $\mathbf{L_0 = 645\ mm}$ measured from the saddle to the nut, the first fret $\mathbf{L_1}$ should be placed at a distance of

$$\mathbf{L_1} = \left(\frac{17}{18}\right)(645\text{mm}) = 609.17\text{mm}.$$

The second fret must now be placed at a distance that is $\frac{17}{18}$ th of the distance to the first fret. In our numerical example we have,

$$\mathbf{L_2} = \left(\frac{17}{18}\right) 609.17\text{mm} = \left(\frac{17}{18}\right)\left(\frac{17}{18}\right)(645\text{mm}) = 575.32\text{mm}.$$

The third fret will have a position given by,

$$\mathbf{L_3} = \left(\frac{17}{18}\right) 575.32\text{mm} = \left(\frac{17}{18}\right)\left(\frac{17}{18}\right)\left(\frac{17}{18}\right)(645\text{mm}) = 543.36\text{mm}.$$

We may now generalize to the nth fret given a fixed open string length of $\mathbf{L_0 = 645\ mm}$, the ***exponential model*** will be given by,

$$\mathbf{L_n} = L(n) = (645\text{mm})\left(\frac{17}{18}\right)^{\mathbf{n}}$$

where n is the input variable corresponding to the fret number. The function L(n) has a domain which is the Natural Numbers and is therefore defined discretely.

The following table gives one octave of fret position relative to the saddle for the **A** string of an acoustic guitar (*Kaman Applause model AE-38*). Each fret position is correlated to the associated musical note. The table is generated using the preceeding models and taking an open string length $\mathbf{L_0}$ of 645mm.

Fret Placement based on the Model

$$\mathbf{L(n) = L_0 \left(\frac{17}{18}\right)^n}$$

note in the scale	corresponding fret position	relative predicted distance from saddle in millimeters
A	0	645.00
$A^{\#}$	1	609.17
B	2	575.32
C	3	543.36
$C^{\#}$	4	513.17
D	5	484.67
$D^{\#}$	6	457.74
E	7	432.31
F	8	408.29
$F^{\#}$	9	385.61
G	10	364.19
$G^{\#}$	11	343.95
A	12	324.85

Notice that at the twelfth fret the string length should be half the overall length in order to correspond to twice the frequency.

The Modern Exponential Model for Guitar Fret Placement

The contemporary model for guitar placement is based on the physical principle that the length of a vibrating string varies inversely with frequency. Hence, applying the musical system found in western music we have

$$L = \frac{k}{f} = \frac{k}{f(n)} = \frac{k}{f_0(2)^{n/12}} = \frac{k}{f_0}(2)^{-n/12} .$$

Now by defining

$L_0 = \frac{k}{f_0}$ as the length of the guitar string,

n is the fret number,

L(n) is the length of the string as a function of the nth\ fret.

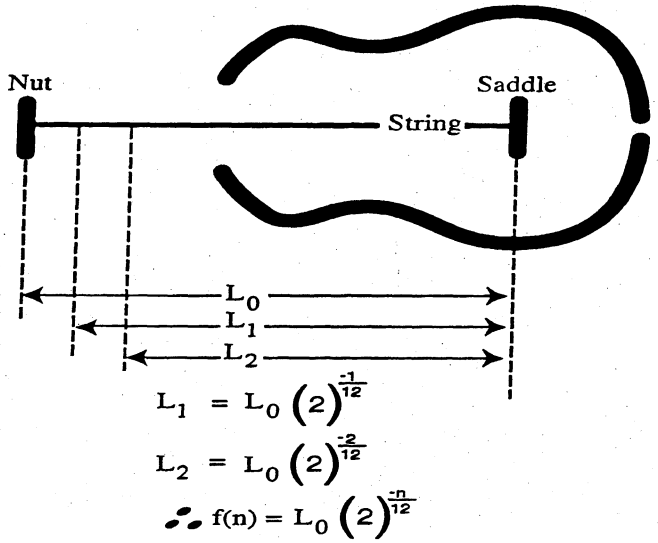
For a guitar string of open length L_0

$$L(n) = L_0 (2)^{-n/12} ,$$

which defines fret placement relative to the saddle of the instrument.

The following table gives one octave of fret position relative to the saddle for the A string of an acoustic guitar (*1978 Yamaha C-200*). Each fret position is correlated to the associated musical note. The table is generated using the preceding models and taking an open string length L_0 of 645mm.

Modern Fret Placement



Fret Placement based on the Exponential Decay Model

$$L(n) = L_0\left(\frac{1}{2}\right)^{n/12}$$

enter as $y_1 = 645*(1/2)\ \hat{}\ (x/12)$ and complete the following table

note in the scale	corresponding fret position	relative predicted distance from saddle in millimeters
A	0	645.00
$A^{\#}$	1	608.80
B	2	574.63
C	3	542.38
$C^{\#}$	4	511.94
D	5	483.20
$D^{\#}$	6	456.08
E	7	430.49
F	8	406.32
$F^{\#}$	9	383.52
G	10	361.99
$G^{\#}$	11	341.68
A	12	322.50