LINKING NUMERIC AND GEOMETRIC REPRESENTATIONS OF NUMBER THEORY CONCEPTS IN A SPREADSHEET ENVIRONMENT

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Introduction

Graphing capabilities of spreadsheets have been used in the preparation of mathematics teachers mostly for constructing functional relationships between two variables with respect to a set of coordinate axes [1-3]. Although the use of a spreadsheet as a generator of geometric figures from numerical tables has been known in the context of advanced mathematics, computer, and engineering education for about a decade [4], it has been less generally recognized in the context of mathematics teacher education. Yet, such a utilization of the software in the preparation of teachers seems to be didactically sound for, among other things, it puts the ancient tradition of the geometrization of mathematical structures in the context of contemporary educational discourse.

This paper shows how the graphing capabilities of spreadsheets enable interactive geometric constructions from the results of numerical modeling of several fundamental concepts in number theory presented in the form of homogeneous Diophantine equations of the second order. To this end, the paper focuses on the systematic construction of integral solutions to classic equations of that type in the form of numerical tables and interactive geometrization of such solutions in the form of triangles. One didactical significance of such use of a spreadsheet is to introduce several number theory concepts as problem solving tools of computational environments designed to be interactively mapped into the geometric domain.

Integral solutions to second degree algebraic equations

The problem of finding all integral solutions to a second-degree polynomial equation in three unknowns with integral coefficients belongs to a classic branch of the elementary theory of numbers with a long and fascinating history. In geometric terms, two special cases of such an equation represent relationships between integral sides x, y, and z of a right triangle and scalene triangle with a 60° angle respectively

$$x^2+y^2=z^2 \quad (1)$$

$$x^2-xy+y^2=z^2 \quad (2)$$

A set of three integers satisfying equation (1) is called a Pythagorean triple and (3, 4, 5) is the simplest example of such a primitive P-triple. (The triple (x, y, z) is said to be primitive if the greatest common divisor of x, y, and z is 1). Cuoco [5] suggested to call a set of three integers satisfying equation (2) an Eisenstein triple; one can check that (5, 7, 8) is a primitive E-triple.

Also, one can consider the following generalization of the above equations

$$x^2-2xy\cos\gamma+y^2=z^2 \qquad (3)$$

A triple of integers (x, y, z) satisfying equation (3) will be referred to below as a γ-triple. Thus a P-triple is a 90°-triple and an E-triple is a 60°-triple. The following questions, both technological and pedagogical, arise at this point:

- How can primitive γ-triples, including the above mentioned special cases, and their associated triangles (i.e., geometrizations of the triples) be generated within a spreadsheet?
- What conceptual insights about different triples and relationships among them can one gain in a numeric environment of a spreadsheet?
- What mathematical concepts different from triples can be introduced through the process of construction of the computational environments in question?
- How do such environments support one's ability to experiment with mathematical context, generate new knowledge about triples, and make mathematical connections?
- How does a spreadsheet-enabled experimentation with and geometrization of number theory concepts support one's understanding of the concepts?

This paper is aimed to address the above questions. It is motivated by work done with pre-service teachers of mathematics with special emphasis on the use of a spreadsheet as tool kit [6,7]. The metaphor of a tool kit in the context of technology-enabled mathematics instruction means an array of representational formats that mediate one's mathematical thinking in a technology-rich environment. A tool kit approach to the teaching and learning of mathematics in a spreadsheet environment is based on the assumption that the variety of qualitatively different representational formats (notation systems) provided by the environment differently affects the acquisition of new concepts by learners. A spreadsheet-based tool kit includes graphic, geometric, iconic, numeric, and other types of representational formats. Some of these representations are used in this paper to illustrate the approach.

Generating 90°-triples through the Euclidean algorithm

The general formulas for 90°-triples have been well known since the time of Euclid:

If (x, y, z) is a primitive Pythagorean triple, then one of x and y is even, and the other is odd. If y is even, then

$$x=m^2-n^2,\ y=2mn,\ z=m^2+n^2 \qquad (4)$$

where m and n are relatively prime positive integers of opposite parity, m>n.

Formulas (4) can be utilized in a computational environment for generating non-trivial solutions to equation (1). This environment should be capable of coordinating two related arithmetical properties of the generators *m* and *n*. This coordination requires the use of the Euclidean algorithm.

An educative value of this computational environment is not only in the systematic presentation of all primitive 90°-triples related to a given generator *m*, but also in the fact that its structure invites new observations and stimulates new inquiries. For example, for a given *m*, how many *n* exist? In other words, how many 90°-triples can be generated within the environment? Through modeling 90°-triples on a spreadsheet, the Euler phi-function $\varphi(m)$ defined as the number of positive integers not greater than and relatively prime to *m* can be introduced in applied context. In terms of this function, one can come up with the following computationally motivated conjecture: *For a given generator m, there are $\varphi(m)$ generators n, when m is even, and $\varphi(m)/2$ generators n, when m is odd.*

Generating γ-triples within a spreadsheet

Consider the following formulas [8] which provide all integral solutions to equation (3):

$$x=m^2+2mn;\ y=2mn+2(1-\cos\gamma)n^2;\ z=m^2+2(1-\cos\gamma)(mn+n^2) \qquad (5)$$

where *m* and *n* are relatively prime integers. In particular,

for $\gamma=90°$, $(x, y, z)=(m^2+2mn,\ 2mn+2n^2,\ m^2+2mn+2n^2)$;

for $\gamma=60°$, $(x, y, z)=(m^2+2mn,\ 2mn+n^2,\ m^2+mn+n^2)$; and

for $\gamma=120°$, $(x, y, z)=(m^2+2mn,\ 2mn+3n^2,\ m^2+3mn+3n^2)$.

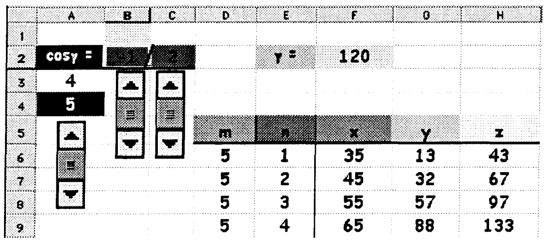
Apparently, other integer values of the angular parameter γ would not produce formulas for rational, let alone integer triples. In order to generate integer γ-triples for rational values of $\cos\gamma$ (different from 1/2, -1/2 and 0) one has to substitute in (5) p/q for $\cos\gamma$ and use the formulas

$$x=q(m^2+2mn),\ y=q[2mn+2(1-p/q)n^2],\ z=q[m^2+2(1-p/q)(mn+n^2)] \qquad (6)$$

for the construction of integer γ-triples, where *p* and *q* are relatively prime integers.

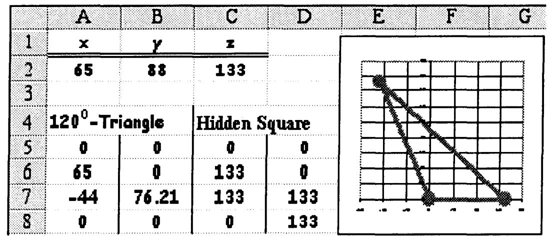
Formulas (6) do not guarantee the triples to be primitive though, even if *m* and *n* as well as *p* and *q* are relatively prime respectively. Indeed, the case of m=6, n=1, p=2, and q=5 yields the triple (240, 66, 222) which is not primitive. In order to generate primitive γ-triples only, once again, one can incorporate the Euclidean algorithm into the spreadsheet. This algorithm divides each element of a γ-triple constructed via formulas (6) by their greatest common divisor.

Figure 1 shows such a spreadsheet that generates 120°–triples. Formulas (6) are incorporated in the range **F6:H6** and replicated down this range. The Euclidean algorithm is used by the spreadsheet three times. First, like in the case of 90°-triples, the algorithm enables generators *m* and *n* be relatively prime with the weakened condition regarding their different parity. Two other applications (along with the first hidden to the right of column **H**), enable 120°–triples to be relatively prime.



	A	B	C	D	E	F	G	H
1								
2	cosγ =				γ =	120		
3	4							
4	5							
5				m	n	x	y	z
6				5	1	35	13	43
7				5	2	45	32	67
8				5	3	55	57	97
9				5	4	65	88	133

Figure 1. Generation of 120°–triples.



	A	B	C	D
1	x	y	z	
2	65	88	133	
3				
4	120°-Triangle		Hidden Square	
5	0	0	0	0
6	65	0	133	0
7	-44	76.21	133	133
8	0	0	0	133

Figure 2. Geometrization of a 120°–triple.

Geometrization of γ-triples

In general, one can represent γ-triples geometrically using a spreadsheet. To this end, the numeric environment that generates γ-triples according to formulas (6) can be connected to numeric domain using the idea described in the previous section. More specifically, one can see that if γ=120°, infinitely many 120°-angled triangles whose side lengths are relatively prime integers can be generated. However, for each generator *m* there exists finite number of such triangles; one such a triangle has relatively prime integer sides 13, 35, 43, satisfying the law of cosine with the largest side being opposite to 120° angle. By setting the table of four basic points (0, 0), (x, 0), (ycosγ, ysinγ), (0, 0) and plotting them on the XY-Scatter selection from the Chartwizard menu, one can construct the edges of the triangle corresponding to a *γ*-triple (Figure 2).

In addition, two new columns with the second series of points (0, 0), (p, 0), (p, p), (0, p), where p=max(x, y, z), has to be added to the table in order to produce an accurate portrayal of the angle γ. The 120°-triple (65, 88, 133) and its geometric representation in terms of an obtuse triangle are shown in Figures 1 and 2 respectively. The additional data that generates a square is hidden from view in Figure 2.

The occurrence of sister γ-triples

Using a spreadsheet as a generator of γ-triples one may come across the following 'sister' 60°-triples with two common elements, (8, 3, 7) and (8, 5, 7), that both satisfy equation (3). Several questions arise at this point: (i) How can the existence of sister 60°-triples, or more generally γ-triples, be interpreted in algebraic terms? (ii) How can the spreadsheet-enabled geometrization of γ-triples provide an interpretation of this phenomenon?
(iii) For what values of γ do sister γ-triples occur?

One may note that both triples (x, y, z) and (x, x-y, z) satisfy equation (3) and this observation gives a general rule for generating sister 60°-triples. Does this rule generate all sister 60°-triples? The simultaneous geometrization of the triples (8, 3, 7) and (8, 5, 7) shown in Figure 3 gives a geometric interpretation of this phenomenon. Apparently, this interpretation holds true for values of γ different from 60°, yet it suggests that no sister γ-triples exist for $\gamma \geq 90°$.

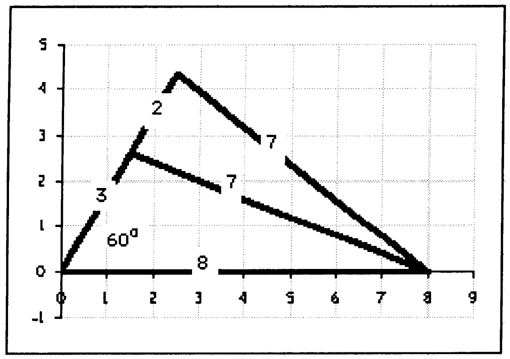


Figure 3. Spreadsheet-enabled geometrization of sister triples.

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