Modeling and Applications with LaGrange Multipliers using MAPLE

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We begin our discussion of optimization, modeling, and MAPLE with equality constrained problem from multivariable calculus. We present the graphical interpretation first with both contour plots and 3-D plots. Next, we illustrate the use of MAPLE to solve the set of partial derivatives leading to a solution. We interpret the solution in context of an application' example.

Graphical Interpretation

Maximize $z = -2x^2 + -y^2 + xy + 8x + 3y$
s.t. $3x + y = 6$

We obtained a contour plot of z from MAPLE and overlaid the single constraint onto the contour plot, see Figure 1. Let's see what information can we obtain from this graphical representation. First, we note that the unconstrained optimal does not lie on the constraint. We can estimate the unconstrained optimal $(x^*,y^*) = (2.3,1.3)$ as the approximate center of the inner most contour.. The optimal *constrained* solution lies at the point where the constraint is tangent to a contour of the function, *f*. This point (let's call it X*) is estimated to be about (1.8,1.0). Clearly, we see that the resource does not pass through the unconstrained maximum and thus, improvements can be made in the solution if we can afford more resource. Through more resource, the line will move until we get the line to pass through the unconstrained solution. At that point, we would no longer add (or subtract) any more resources (see Figure 2). We can gain valuable insights about the problem if we are able to plot the information and experiment with the constraints.

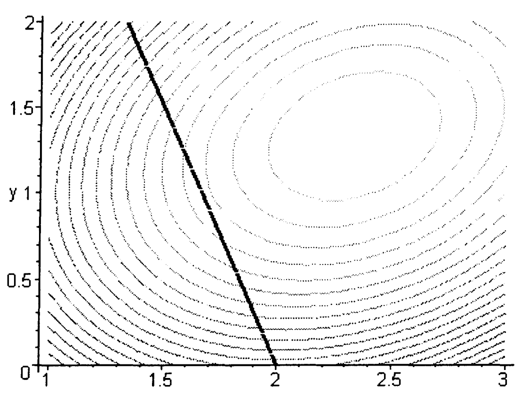

Figure 1. Contour plot of equality constraint example g(x)= 3x + y =6.

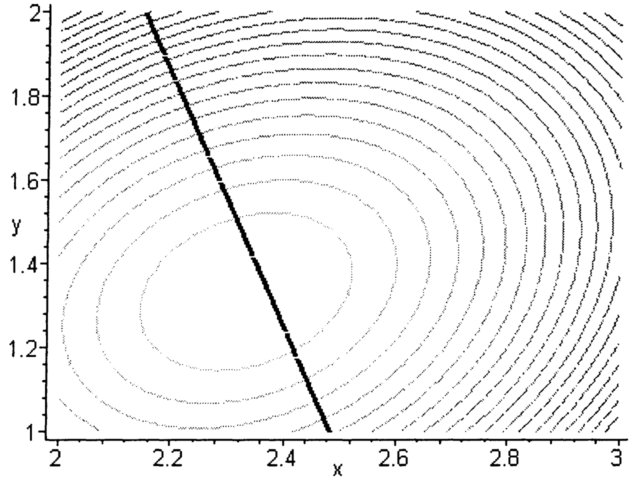

Figure 2. Adding more resource, $g(x) = 8.45 = 3x+y$

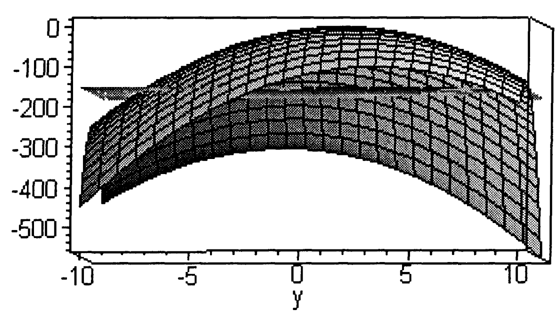

Figure 3. The 3-D plot

Computational Method of LaGrange Multipliers with Maple

You are employed as a consultant for a small oil transfer company. The management desires a minimum cost policy due to the restricted tank storage space. Historical records have been studied and a formula has been derived that describes system costs:

$$f(\mathbf{x}) = \sum_{n=1}^{N} (A_n B_n) / X_n + (H_n X_n) / 2$$

where:

A_n is the fixed costs for the nth item.

B_n is the withdrawal rate per unit time for the nth item.
H_n is the holding costs per unit time for the nth item.

The tank space constraint is given by:

$$g(\mathbf{x}) = \sum_{n=1}^{N} t_n X_n = T$$

where: t_n is the space required for the nth item (in correct units)
T is the available tank space (in correct units)

You determine the following information:

Item(n)	A_n (\$)	B_n	H_n (\$)	t_n (cubic feet)
1	9.6	3	0.47	1.4
2	4.27	5	0.26	2.62
3	6.42	4	0.61	1.71

You measure the storage tanks and find that there is only 22 cubic feet of space available. Find the optimal solution using a *minimum cost policy.*

First, solve the unconstrained problem.

If we assume that $\lambda = 0$, we first find an unconstrained optimal solution.

```
> Lf:=  (9.6)*(3)/x + .47*x/2 + (4.27)*(5)/y + .26*y/2 + (6.42)*(4)/z +
.61*z/2;
```

$$Lf := 28.8\,\frac{1}{x} + .2350000000\,x + \frac{21.35}{y} + .1300000000\,y + \frac{25.68}{z} + .3050000000\,z$$

```
> pd1:=diff(Lf,x);
```

$$pd1 := -28.8\,\frac{1}{x^2} + .2350000000$$

```
> pd2:=diff(Lf,y);
```

$$pd2 := -21.35\,\frac{1}{y^2} + .1300000000$$

```
> pd3:=diff(Lf,z);
```

$$pd3 := -25.68\,\frac{1}{z^2} + .3050000000$$

```
> solve({pd1=0,pd2=0,pd3=0},{x,y,z});
```

$\{z = -9.175877141, y = -12.81525533, x = -11.07037450\},$
$\{x = 11.07037450, z = -9.175877141, y = -12.81525533\},$
$\{z = 9.175877141, y = -12.81525533, x = -11.07037450\},$
$\{z = 9.175877141, x = 11.07037450, y = -12.81525533\},$
$\{y = 12.81525533, z = -9.175877141, x = -11.07037450\},$
$\{x = 11.07037450, y = 12.81525533, z = -9.175877141\},$
$\{z = 9.175877141, y = 12.81525533, x = -11.07037450\},$
$\{z = 9.175877141, x = 11.07037450, y = 12.81525533\}$

The only useful solution from the above solutions is where each of the variables *x, y,* and *z* are greater than or equal to zero.

x= 11.07037450, y= 12.81525533, z= 9.175877141

This unconstrained solution is (x*,y*,z*) = (11.07, 12.82, 9.176). This solution provides an upper bound since those values will not satisfy the constraint,
1.4 x+ 2.62 y + 1.71 z = 22.

The constrained solution is found by the following methodology:
1) Set up the LaGrangian function, L.
Let x = item 1
y= item 2
z= item 3

$L(x,y,z,\lambda) = (9.6)(3)/x + .47x/2 + (4.27)(5)/y + .26y/2 + (6.42)(4)/z + .61z/2 + \lambda\,[\,1.4\,x + 2.62\,y + 1.71\,z - 22]$
2) Find all the partial derivatives set equal to zero.

$L_x = -28.8\,x^{-2} + .235 + 1.4\,\lambda = 0$

$L_y = -21.35\,y^{-2} + .13 + 2.62\,\lambda = 0$

$L_z = -25.68\,z^{-2} + .305 + 1.71\,\lambda = 0$

$L_\lambda = 1.4\,x + 2.62\,y + 1.71\,z - 22 = 0$

```
> L:=  (9.6)*(3)/x + .47*x/2 + (4.27)*(5)/y + .26*y/2 + (6.42)*(4)/z +
.61*z/2 + l1 *( 1.4* x + 2.62* y + 1.71* z -22);
```

$$L := 28.8\frac{1}{x} + .2350000000\,x + \frac{21.35}{y} + .1300000000\,y + \frac{25.68}{z} + .3050000000\,z + l1\,(1.4\,x + 2.62\,y + 1.71\,z - 22)$$

```
> nc:=grad(L, [x,y,z,l1]);
```

$$nc := \left[-28.8\frac{1}{x^2} + .2350000000 + 1.4\,l1,\ -21.35\frac{1}{y^2} + .1300000000 + 2.62\,l1,\ -25.68\frac{1}{z^2} + .3050000000 + 1.71\,l1,\ 1.4\,x + 2.62\,y + 1.71\,z - 22\right]$$

3) Solve the set of partial derivatives .

```
> solve({-28.8*1/(x^2)+.2350000000+1.4*l1, -
21.35*1/(y^2)+.1300000000+2.62*l1, -25.68*1/(z^2)+.3050000000+1.71*l1,
1.4*x+2.62*y+1.71*z-22},{x,y,z,l1});
```

$$\{y = 3.213131453, z = 4.044536153, x = 4.761027695, l1 = .7396771332\}$$

```
> subs({y = 3.213131453, z = 4.044536153, x = 4.761027695, l1 =
.7396771332},L);
```

$$21.81316118$$

4) Check to see if we found a maximum or minimum value.
Do we have the minimum?
The Hessian matrix, H is:

```
> h:=hessian(L, [x,y,z]);
>
```

$$h := \begin{bmatrix} 57.6\frac{1}{x^3} & 0 & 0 \\ 0 & 42.70\frac{1}{y^3} & 0 \\ 0 & 0 & 51.36\frac{1}{z^3} \end{bmatrix}$$

```
> h1:=det(h);
```

$$h1 := 126320.9472\frac{1}{x^3\,y^3\,z^3}$$

```
> subs({y = 3.213131453, z = 4.044536153, x = 4.761027695, l1 =
.7396771332},h1);
```

$$.5333123053$$

The Hessian Matrix is positive definite at (x*,y*,z*)= (4.5, 3, 3.82). Therefore, the solution found is the minimum for this convex function with a linear equality constraint.
5) Interpret the shadow price, λ.

Should we add storage space? We know from the unconstrained solution that, if possible, we would add storage space to decrease the costs. Additionally, we have found the value of λ was 0.85, which suggests that any small increase (Δ)in the RHS of the constraint causes the objective function to decrease by approximately .85Δ. Thus, the total cost of adding an extra storage tank would have to be less than the savings incurred by adding the new tank.