ENERGIZING MULTIVARIABLE CALCULUS VIA MAPLE®

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INTRODUCTION

Finding a general consensus about whether to use technology in the mathematics classroom or not is a hopeless task; most educators fall somewhere between the two extremes on that spectrum. Yet there is one subject that should garner overwhelming support for technology-based pedagogy: multivariable calculus. Visualizing three-dimensional objects is so effortless with a computer algebra system (CAS) that it is hard to envision teaching without a computer.

MAPLE® PRIMER FOR MULTIVARIABLE CALCULUS

The most typical complaint that students have about using a CAS in mathematics is the onerous task of learning the commands and syntax. Despite this criticism, having a command of the software's language is vital. Otherwise, we fall into the trap of "push button" mathematics where students mindlessly alter pre-existing templates.

There are several strategies which can help an instructor counter this complaint and make the learning experience of using a CAS as seamless as possible. First, the syntax should be demonstrated in regular, everyday class meetings – preferably with students at their own computers for experimentation. Second, students need an easy-to-follow resource.

The author's Maple® primer for multivariable calculus is free for classroom use, provided that it is distributed (in print or electronic form) in its entirety with the author's information clearly displayed at the top. This primer is too long to be reproduced here, but it is available for download as a web page, Microsoft Word® document, or Maple® worksheet.

From the author's home page (http://www.facstaff.oglethorpe.edu/jnardo/), click the "Academic Research, Publications, and Presentations" hyperlink on the right, scroll to the ICTCM 2002 information, and click the "Handouts and Materials" hyperlink.

CLASSROOM-READY PROJECTS

Five projects are available at this same Internet location. As a further aid, high school and collegiate faculty may e-mail the author (jnardo@oglethorpe.edu) for keys to these projects, provided as Maple® worksheets. The remainder of this article shows three of these classroom projects.

PROJECT TWO – ARC LENGTH AND CURVATURE

Let $\vec{r}(t) = < f(t), g(t), h(t) >$ be vector function in $\mathbb{R}^3$ with domain $[a,b]$.

The curvature function $\kappa(t)$ of this space curve is constructed via the following formula:

$$\kappa(t) = \frac{\left|\vec{r}\,'(t) \times \vec{r}\,''(t)\right|}{\left|\vec{r}\,'(t)\right|^3}.$$

The arc length of this space curve is calculated via the following definite integral:

$$arc\,length = \int_a^b \left|\vec{r}\,'(t)\right| dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt.$$

1. Consider the vector function $\vec{p}(t) = < \cos(t), \sin(t), 0 >$ with domain $[0, 2\pi]$.

(A) Define p as a vector function. Sketch a graph of this space curve.

(B) Differentiate each coordinate function of p separately using Maple. Combine to make the velocity/first derivative vector. Define it as a vector function with the name pvel.

(C) Differentiate each coordinate function of pvel separately using Maple. Combine to make the acceleration/second derivative vector. Define it as a vector function with the name pacc.

(D) Calculate the cross product of pvel and pacc. Simplify. Define this simplified vector in Maple as pcross.

(E) Calculate the length of pcross. Simplify.

(F) Calculate the length of pvel. Cube it. Then simplify.

(G) Construct the curvature function. Define it as an ordinary function in Maple named pkap.

(H) Sketch the graph of the curvature function using the plot command. What does this graph indicate about the curvature of the original curve?

(I) Calculate the arc length of this curve using calculus methods.

(J) Why is the answer from part (I) not a surprise?

2. Consider the vector function $\vec{r}(t) = < t\cos(t), t\sin(t), t >$ with domain $[0, 2\pi]$.

(A) Using the level of detail from #1 as a guide, construct and graph the curvature function for r.

(B) From the curvature graph, estimate the value of the parameter t where the curvature is maximum.

(C) What is the corresponding point on the curve in 3-space where the curvature is maximum?

(D) What is this maximum curvature? A numerical approximation is acceptable.

(E) Calculate the arc length of this curve using calculus methods. A numerical approximation is acceptable.

PROJECT THREE – PARTIAL DERIVATIVES

Consider the following multivariable function: $f(x,y) = \dfrac{-5x}{x^2 + y^2 + 1}$.

You are interested in the behavior of this function near the point (– 1, +1).

(A) Define this function in Maple. Evaluate this function at the point (– 1, +1).

(B) Use the plot3d command to graph this function on the domain $[-3,3]\times[-3,3]$. Color this surface yellow and name it "graph."

(C) Use the implicitplot3d command to graph the plane $y=1$ on $[-3,3]\times[-3,3]\times[-2,2]$. Color this plane black and name it "plane1."

(D) Display these two graphs together.

(E) These two mathematical objects intersect in a space curve: $<t,1,f(t,1)>$. Use the space curve command to graph this curve on the domain $-3 \le t \le 3$. Color this curve red and name it "curve1."

(F) Display the function's graph along with plane1 and curve1.

(G) Use the implicitplot3d command to graph the plane $x=-1$. Color this plane black and name it "plane2."

(H) Display the original function's graph and this plane's graph together.

(I) These two mathematical objects intersect in a space curve: $<-1,t,f(-1,t)>$. Use the space curve command to graph this curve on the domain $-3 \le t \le 3$. Color this curve blue and name is "curve2."

(J) Display the original function's graph along with plane2 and curve2.

(K) Calculate the original function's first partial derivative with respect to *x*. Define this as a multivariable function of *x* and *y* in Maple named pdfx.

(L) Calculate the original function's first partial derivative with respect to *y*. Define this as a multivariable function of *x* and *y* in Maple named pdfy.

(M) Evaluate the first partial derivatives at the point (– 1, +1).

(N) Construct the tangent line to curve1. Define it in Maple as a vector function of *t* named line1.

(O) Construct the tangent line to curve2. Define it in Maple as a vector function of *t* named line2.

(P) Use the spacecurve command to graph both tangent line functions <u>separately</u>. Name these graphs tanline1 and tanline2, respectively.

(Q) Display these tangent line graphs, their respective curves, and the original graph together in one picture. Rotate so that the tangent line to curve1 is displayed correctly, i.e. intersecting the surface at the point of tangency. (<u>HINT</u>: You may need to adjust the domains of the tangent lines to have everything "fit.")

(R) Display a second copy of the tangent line functions, their respective curves, and the original graph together in one picture. Rotate so that the tangent line to curve 2 is displayed correctly, as described above in (Q).

PROJECT FOUR – THE LEAST SQUARES LINE FROM STATISTICS

Very often when collecting and analyzing data, a researcher wishes to build a linear model to "fit" the data, i.e. he/she wants a line that passes as closely as possible to the points in the data.

If there are only two data points, then there is exactly one line which passes through those two points. Using algebra, it is a simple task to find the line. If there are more than two points, then it is not very likely that all the points lie exactly on a line, and there are infinitely-many lines that could be used. We must find the one line that has the best "fit."

One model that is traditionally used by statisticians is called the least squares line. There are formulas for this line in any introductory statistics book; more importantly, these formulas were created by solving an optimization problem using multivariable calculus. You will solve one specific, <u>small</u> example of finding a least squares line for a data set.

Consider the set of points: $\{(1,1)\ ,\ (2,4)\ ,\ (3,9)\ ,\ (4,9)\}$.

1. Enter this set of points as a list named DATA in Maple. Since there is more than one point, you will use the curly braces as above. Maple, however, uses square brackets to represent the individual points, i.e. [1,1] and not (1,1).

2. Load the plots package. Use the pointplot command to draw a scatterplot named DOTPLOT of these data points. Display the scatterplot by itself.

3. One line which could be used as a possible model for these data is: $y = 5x - 6$. Create a plot named LINE1 which displays this linear function on the domain $[0,5] \times [0,10]$. Display the scatterplot and this line together. How many data points are on the line? How many are above the line? How many are below the line?

4. A second line which could be used as a possible model for these data is: $y = \frac{5}{2}x - 1$. Create a plot named LINE2 which displays this linear function on the domain $[0,5] \times [0,10]$. Display the scatterplot and this line together. How many data points are on the line? How many are above the line? How many are below the line?

5. Which of these two possible models for the data do you prefer? Why?

We will now find the least squares line – the line of best fit for this data set. Consider the generic linear function: $y = L(x) = mx + b$. We will calculate the vertical "error" between the exact data points and the points predicted by our generic linear function above. Define L as a function in Maple: >L:= x → m*x+b;

6. What is the height of the function $L(x)$ above the data value $x = 1$, i.e. $L(1)$? What is the difference/error between this height and the height (y) given by the data set for $x = 1$? Note that your difference will involve the variables m and b.
7. Repeat #6 for $x = 2$, $x = 3$, and $x = 4$.

Some of the differences from #6 and 7 will be positive and some will be negative, depending on the eventual choice of m and b. We do not wish for there to be any cancellation; thus, we will square each of these differences to eliminate any minus signs from happening.

8. Square each of the differences from #6 and 7 and add them up. This will result in an expression involving m and b. Define this as a multivariable <u>function</u> named E: > E:= (m,b) → some rule involving m and b.

E represents the total squared error in representing the data by the line $y = L(x) = mx + b$. We want to minimize the error function E(m,b) to give the least possible error in fitting our generic line to the data set. We may use any real number for m and any real number for b; thus, the domain of E(m,b) is the entire plane $\mathbb{R}^2$.

9. Find all stationary points of E(m,b).
10. Explain why there are no singular points of E(m,b).
11. Explain why there are no boundary points for E(m,b).
12. Find the minimum value for E(m,b). This is the minimum error. Prove this is a minimum using the Second Derivatives Test.
13. At which point in the domain (m,b) does this minimum occur, i.e. which stationary point yields the minimum? This gives a specific choice for m and b.

The specific linear function with the m and b which yield the minimum error is called the least squares line. In statistics, it is used as the line of best fit for a data set (when a line fits the data well).

14. Define the least squares line as a function named L. Recall $L(x) = mx + b$; use the <u>specific</u> m and b found above. Create a plot named LSL which displays this linear function on the domain $[0,5] \times [0,10]$. Display the scatterplot and this line together. How many data points are on the line? Above? Below?
15. Do you feel geometrically that the least squares line "fits" this data well? Explain using what you see in the graph from #14.
16. There is no data in the table for $x = 5$. Use the least squares line to predict the height/function value for $x = 5$.

HOW TO PLOT THE LIST OF POINTS: (0,0), (1,1), and (2,2) IN MAPLE

```
> with(plots);
> data:={ [0,0],[1,1],[2,2]};
> pointplot(data);
```