On Teaching Limits Using Handheld Technology

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Abstract. In this article a selection of examples is used to illustrate how the numerical and graphical capabilities of graphing calculators can be used to enhance the teaching and learning of limits. The variety of data types available in these tools facilitates the use of multiple numerical approaches. This in turn allows for a more in depth treatment of this topic, and for introducing some of these ideas at a lower level.

The limit of a function is one of the fundamental concepts in the study of calculus. It has also been traditionally one of the hardest topics for students to grasp conceptually. The use of technology, in particular of modern graphing calculators, facilitates, without a major time investment, using the numerical and graphical approaches to complement the algebraic approach enhancing the teaching and learning of this concept. Some basic capabilities of these machines make this possible. First of all, the speed and precision of these inexpensive machines facilitate the use of estimation as a viable problem solving strategy. Also, as we have began to see in the new curricula [Core-Plus Mathematics Project, 1998], the ability to immediately visualize the graph of functions is changing the way we approach the teaching of many calculus topics. Moreover, the variety of data types that modern graphing calculators provide, allows for the use of multiple approaches to introduce numerical ideas both in precalculus and in calculus. Thus, it is possible to study key calculus concepts in the way they were developed and are better understood, that is, as limits of approximations. Finally, we will see that graphing calculators have the potential for increasing the scope and the degree of difficulty of the problems that can be posed.

Our goal in this article is to review some basic problems to illustrate how the numerical and graphical capabilities of graphing calculators can be used to enhance the teaching of limits. The problems presented were selected to represent different approaches, accessible at various levels, to ideas related to the concept of limits.

The syntax used in the commands and the screens provided corresponds to the Texas Instruments TI-83. The screens included, sometimes in excess, will remove any doubts in reproducing the solutions provided. Our initial work was influenced by two of the pioneer works on the integration of technology [Demana & Waits, 1993; Dick & Patton, 1994].

Exploring the local and global behavior of a function

In these first examples we illustrate the graphical approach together with the numerical approach that will be presented using tables, sequences and lists, and recursion.

Example. Use numerical and graphical evidence to explore the behavior of the following functions at the given values, I) $f(x)=\dfrac{\sin x}{x}$ near $x=0$; II) $h(x)=\begin{cases} x^2, x\le 1 \\ 3-2x, 1<x<2 \\ x/4+1, 2\le x \end{cases}$, near $x=2$; III) $g(x)=\dfrac{|x+1|}{|\sqrt{x+2}-1|}$ near $x=-1$; and IV) the behavior of $g(x)=(1+1/x)^x$ as x grows unbounded. Compare your answer with ***e***.

Solution. The first numerical approach shows that the speed of modern graphing calculators makes the table feature a simple and yet powerful tool for analysis, which is perfectly accessible in precalculus [ ].

I) Let $y_1=\dfrac{\sin x}{x}$ and select *Ask* for the independent variable in *Table Setup*. Then, giving to x values arbitrarily close to zero, first from the right and then from the left, we see as illustrated in figure 1.a, that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$. The graph (figure 1.b) can be traced, after zooming in if necessary, to confirm visually the result and the independence of the existence of the limit from the value of the function at $x=0$. Notice that the symmetry of the graph with respect to the y-axis can be easily read from the table, as well as from the graph.

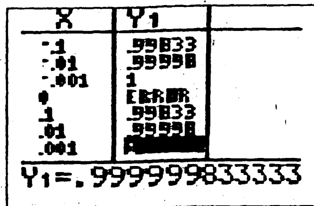

Figure1.a

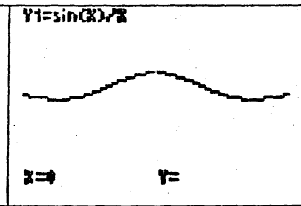

Figure 1.b

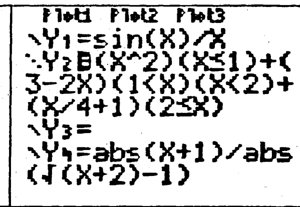

Figure 1.c

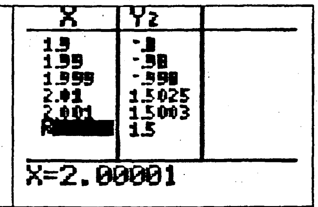

Figure 1.d

II. Figure 1.c illustrates the syntax needed for the piecewise defined function $h(x)$. The table of figure 1.d shows that the left and right limits converge to different values, a fact that can be visualized and confirmed by tracing on the graph in figure 1.e using *dot* mode.

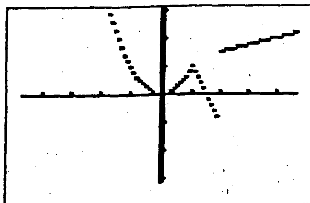

Figure 1.e

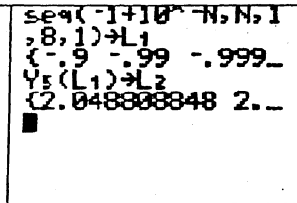

Figure 1.f

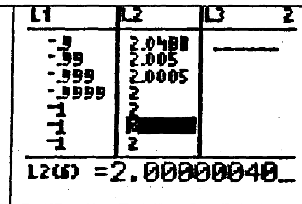

Figure 1.g

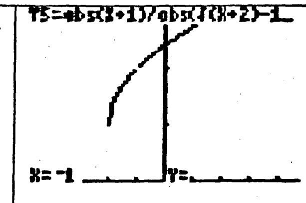

Figure 1.h

III. In this example a second numerical approach is shown. This approach is more sophisticated since it requires the use of sequences and lists. However, the variety and importance of the applications accessible in calculus by using the sequence data structure justify the initial investment of time. We have used this approach successfully with first-year calculus students after they have gained enough familiarity with the table. As seen in figure 1.f, we create two sequences that converge to p from either side using $p \pm 10^{-n}$ for increasingly large values of n. Then, the function $g(x)$ is evaluated at each sequence. Labeling the lists obtained allows using the list editor (figure 1.g) to see the sequences. The results confirm that as x gets closer to -1 from either side, the function approaches *2*. Finally, instead of generating the sequence of values of $g(x)$ after the sequence of x-values is obtained, we can use recursion to generate the sequence of points. Figures 1.i and 1.j show that the ability to concatenate commands using colon, allows us to recursively evaluate $g(x)$ after each new x-value is generated.

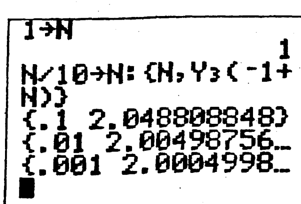

Figure 1.i

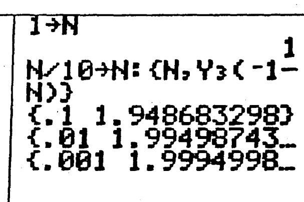

Figure 1.j

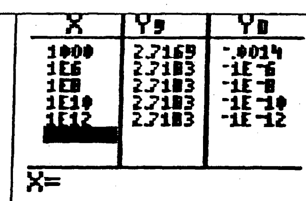

Figure 1.k

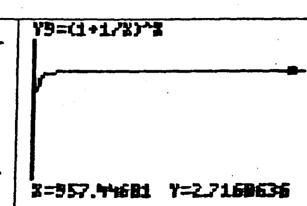

Figure 1.l

IV. As seen in figure 1.k, giving increasingly larger values to x the function seems to converge to an irrational value. To confirm that the function approaches e as x grows unbounded, it suffices to observe the corresponding values of $y_0 = e-(1+1/x)^x$ approach *0*.

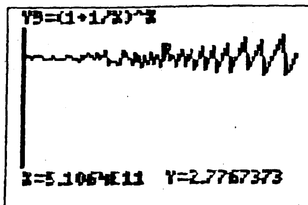

Figure 1.m

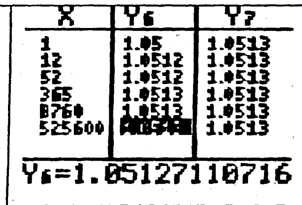

Figure 1.n

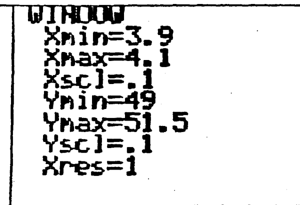

Figure 2.a

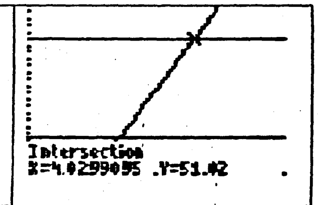

Figure 2.b

Tracing over the graph (figure 1.l) reinforces visually the process. A word of caution, as seen in figure 1.m, if the domain of the function exceeds the precision of the calculator, and students' curiosity guaranties that this will happen, the classical truncation error emerges producing the graph depicted in figure 1.m.

Figure 1.n shows a classical precalculus application used to illustrate how the compound interest of \$1 deposited at a rate of 5% approaches the continuous interest $e^{0.5}$, when the number of compounding periods used in a year increases.

Introducing $\varepsilon-\delta$ problems with an application

Example. We want to cut a disk of area *50.27* in^2 and radius *4 in.* Would it be reasonable to use a blade *0.025* inches thick if we wish to obtain an error in the area of at most *0.75* in^2 ?

Solution. Lets start by defining the functions $y_1 = \pi x^2$, $y_2 = 51.02$, and $y_3 = 49.52$, and then centering the screen about (4,50.27) using a y-radius that allows for the three previous functions to be displayed, and guessing an appropriate small initial radius for the x. As shown in figure 2.a, an initial x-radius of 0.1, and a y-radius of approximately 0.5 were used. The y-radius was big enough to make room for calculations to be displayed at the bottom of the screen. To ensure the tolerance sought, the x-range may have to be modified, until the graph of y_1 enters the screen from the left and exits it from the right, while remaining between y_3 and y_2 as shown in figure 2.d. As illustrated in figure 2.c, the interval *[A, B]*, defined by the x-coordinates of the intersection points of y_1 with y_3 and y_2, respectively, provides the maximum x-range. Finally, the largest possible radius for the x-interval is $\min\{|4-A|,|B-4|\}=0.03$, which shows that the proposed thickness of the blade is acceptable.

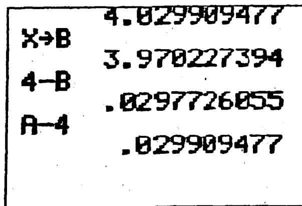	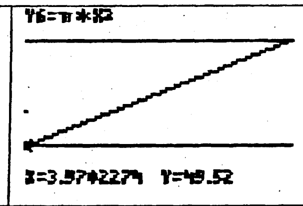	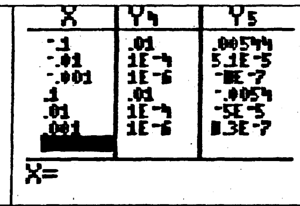	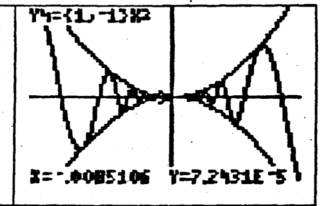
Figure 2.c	Figure 2.d	Figure 3.a	Figure 3.b

Other examples

The behavior of the function $f(x)=x^2\sin(1/x)$ in the vicinity of $x=0$ can be observed visually and numerically in figures 3.a & 3.b, making the sandwich theorem to come alive.

Zooming in the graph of a function such as $g(x)=\sin(\pi/x)$ near $x=0$ or evaluating its values at {1/2, 12/5, 1/3, 2/9, 1/5, 2/13,...}(figure 3.c& 3.d) helps to understand its apparently chaotic behavior.

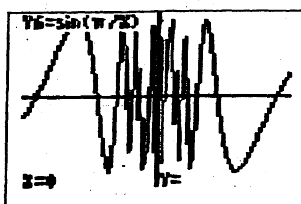	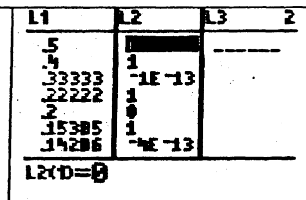	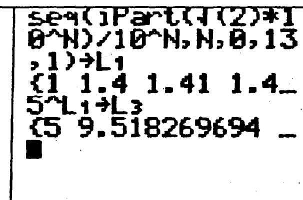	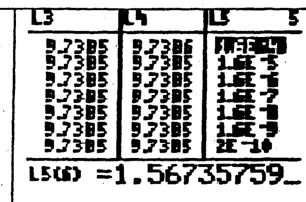
Figure 3.c	Figure 3.d	Figure 4.a	Figure 4.b

Finely, to explore the continuity of $y=5^x$ at $x=\sqrt{2}$, we use the sequences $L_1=\{1.4,1.41,1.414,...\}$ and $L_3=\{1.5,1.42,1.415,...\}$ that approach $\sqrt{2}$ from the left and the right respectively. Hence, evaluating 5^x at L_1 and L_3 we obtain two sequences that approach $5^{\sqrt{2}}$ and whose difference $L_5=L_4-L_3\to 0$. Thus the graph of $y=5^x$ seems to be continuous at $\sqrt{2}$.

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