

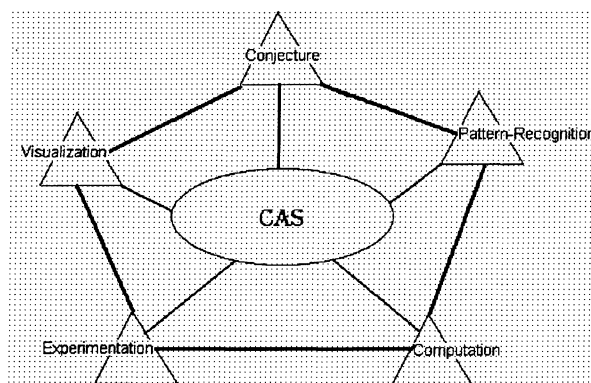
# COMPUTER ALGEBRA SYSTEMS IN MATHEMATICS EDUCATION – COMPUTATION AND VISUALIZATION

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## 1. Five Unifying Elements of a CAS

The impact of computer algebra systems (CAS) on the mathematics curriculum is enormous. The mathematics instructors are gradually changing their teaching techniques because of these technologies (see [1], [2], and [9] ).

Why are CAS such powerful tools to do mathematics? One can identify five essential elements of CAS making them a unique tool in mathematics education. They are the use of a CAS as a tool for **(a) computation (b) visualization (c) experimentation (d) pattern-recognition** and **(e) conjecture-forming**. These are in fact the five essential elements unifying all CAS. Such five-fold usage of a CAS totally empowers mathematics educators and the students at all levels.



**Fig. 1.1 The Five Unifying Elements of a CAS**

The above five unifying elements are however, not at all independent from each other. In fact they are very much interrelated. The full effectiveness of a CAS comes into light when all the elements (a)-(e) are used in harmony with each other.

This paper discusses the unifying elements (a) computation and (b) visualization. Specific examples will be given to describe each role.

We have used *Mathematica* as our choice of the CAS in the following discussion (see [8] and [10]). However, most of these ideas can also be implemented by other CAS such as *Maple* or *Derive*.

## 2. The CAS as a Computational Tool

The CAS are good computational tools because of their vast array of built-in functions. Most of the CAS can simplify, factor, and expand algebraic expressions, solve equations, perform differentiation and integration, solve differential equations, perform matrix calculations, etc. Given below are some of the calculations using *Mathematica*.

### Example 2.1 Solving Equations

Find the exact and approximate solutions of  $x^2 - 2x - 4 = 0$ .

**Input:** `Solve[x^2-2x-4 == 0, x]` (\* Finds the exact solutions\*)  
`NSolve[x^2-2x-4 == 0, x]` (\* Finds the numerical solutions\*)

**Output:**  $\{\{x \rightarrow 1 - \sqrt{5}\}, \{x \rightarrow 1 + \sqrt{5}\}\}$   
 $\{\{x \rightarrow -1.23607\}, \{x \rightarrow 3.23607\}\}$  ■

### Example 2.2 Evaluating Functions via the Functional Notation

Evaluate the function  $f(x) = \frac{x^3 - 4x^2 + 5}{2x + 3}$  at  $x = 2$ ,  $x = -4$ , and  $x = -1/2$ .

**Input:** `f[x_] := (x^3 - 4x^2 + 5)/(2x + 3)` (\*Defines the function f in *Mathematica*\*)  
`{f[2], f[-4], f[-1/2]}` (\*Evaluates f at x = 2, -4, and -1/2 \*)

**Output:** The output is  $\{-3/7, 123/5, 31/16\}$ . ■

### Example 2.3 Difference Quotients, their Limits, and Derivatives

Given  $f(x) = 2x^2 - 3x + 4$ , calculate the difference quotient  $\frac{f(x+h) - f(x)}{h}$  for  $x = 3$  and a suitable number of small  $h$ -values. Calculate  $\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$  and  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ . Also find  $f'(3)$  and  $f'(x)$  directly.

**Input:**  
`f[x_] := 2x^2 - 3x + 4` (\*Defines the function f\*)  
`g[x_] := (f[x + h] - f[x])/h` (\*Defines the difference quotient of f\*)  
`Table[g[3], {h, 0.01, 1.01, 1/20}]` (\*Makes a table of values for g for a fixed x = 3\*)  
`Limit[g[3], h -> 0]`  
`Limit[g[x], h -> 0]`

$f'[3]$  (\*Directly calculates the derivative of  $f$  at  $x=3$ \*)  
 $f'[x]$  (\*Directly calculates the derivative of  $f$  at  $x$ \*)

**Output:**

{9.02, 9.12, 9.22, 9.32, 9.42, 9.52, 9.62, 9.72, 9.82, 9.92, 10.02, 10.12, 10.22, 10.32, 10.42, 10.52, 10.62, 10.72, 10.82, 10.92, 11.02}  
 9  
 $-3+4x$   
 9  
 $-3+4x$  ■

*Are computations useful by themselves?* Perhaps for the engineer or for the researcher getting the final answer could be quite important. However, for the student or for the mathematics educator, this could be the least important aspect of learning mathematics. This is where the visualizations will play a big role. To reap good benefits of a CAS, one must at least combine the computational aspect with the visualization aspect.

### 3. The CAS as a Visualization Tool

Almost all the CAS are fully capable of producing two and three-dimensional graphing. These graphical capabilities make CAS excellent visualization tools (see [3], [4], [5], [6], [7], and [8]). In this section, we will add a visualization aspect to the examples discussed in the previous section.

The following example helps visualize the roots of an equation as the  $x$ -intercepts of the corresponding graph. Also see Example 2.1.

**Example 3.1 Roots,  $x$ -intercepts, and Discriminants**

Discuss the solutions (roots) of the equation  $x^2 - 2x + k = 0$  for  $k$ -values  $-4, 1$  and  $2$ .

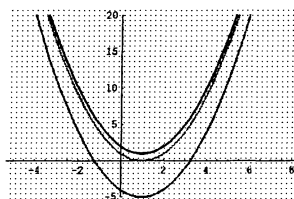
**Input:** Solve[ $x^2 - 2x + k == 0$ ,  $x$ ] /. {{ $k \rightarrow -4$ }, { $k \rightarrow 1$ }, { $k \rightarrow 2$ }}

**Output:** {{{ $x \rightarrow 1 - \sqrt{5}$ }, { $x \rightarrow 1 + \sqrt{5}$ }}, {{ $x \rightarrow 1$ }, { $x \rightarrow 1$ }}, {{ $x \rightarrow 1 - I$ }, { $x \rightarrow 1 + I$ }}}

Depending on the  $k$ -value, the equation has two distinct real roots, one real root, or two distinct non-real roots. To visualize what is happening, one can graph the corresponding functions. Their  $x$ -intercepts will reveal the nature of the roots of the equation for different  $k$ -values. In particular, observe that the  $x$ -coordinates of the  $x$ -intercepts of the graph, if they exist, are nothing but the real roots of the corresponding equation (see Fig. 3.1 below).

**Input:** Plot[Evaluate[ $x^2 - 2x + k$  /. {{ $k \rightarrow -4$ }, { $k \rightarrow 1$ }, { $k \rightarrow 2$ }}, { $x$ ,  $-5$ ,  $8$ }, PlotRange  $\rightarrow$  { $-5$ ,  $20$ }, PlotStyle  $\rightarrow$  {{RGBColor[1, 0, 0], Thickness[1/150]}, {RGBColor[0, 1, 0], Thickness[1/150]}, {RGBColor[0, 0, 1], Thickness[1/150]}}]

**Output:**



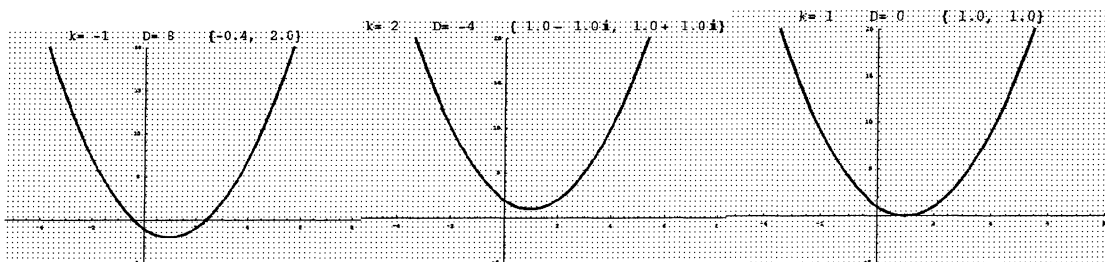
**Fig. 3.1. Roots of  $x^2 - 2x + k = 0$  for  $k = -4, 1$  and  $2$**

The above is a *static visualization*. However, a *dynamic visualization* of the same situation is more effective. The following program creates an animation of the graphs of  $y = x^2 - 2x + k$  for integer  $k$ -values ranging from  $-4$  through  $4$ . The top of each graph displays the current  $k$ -value, the discriminant, and the roots. Observe that when the discriminant is negative, the graph stays clear of the  $x$ -axis, thus not having any real roots. This is a good way to motivate the concept of the **discriminant** of a quadratic equation.

### Program 3.1

```
f[x_] := x^2 - 2x + k
{x1, x2} = x /. NSolve[f[x] == 0, x];
Do[Plot[f[x], {x, -5, 8}, PlotRange -> {-5, 20}, PlotStyle -> {Thickness[1/150], RGBColor[1, 0, 0]},
PlotLabel -> StyleForm[StringForm["k= `` D= `` ``", k, 4 - 4k, PaddedForm[{x1, x2}, {1, 1}]]],
FontSize -> 14, FontColor -> RGBColor[0, 0, 1], FontWeight -> "Bold"],
Epilog -> {PointSize[1/80], RGBColor[0, 0.820325, 0], {Point[{x1, 0}], Point[{x2, 0}]}}, {k, -4, 4}]
```

**Output:** A few frames are given below:



**Fig. 3.2. An Animation of the Graph of  $y = x^2 - 2x + k$  for Different  $k$ -values**

### Example 3.2 Secant Lines, Difference Quotients, Tangent Lines, and Derivatives

We are now in a position to bring the calculations in Example 2.3 into life, using a dynamic visualization! The following program creates an animation of the secant lines to the graph of  $f(x) = 2x^2 - 3x + 4$  through the points  $(3, f(3))$  and  $(3 + h, f(3 + h))$  for changing  $h$ -values. As the increment  $h$  tends to zero, the secant lines will gradually approach the tangent line at  $x = 3$ .

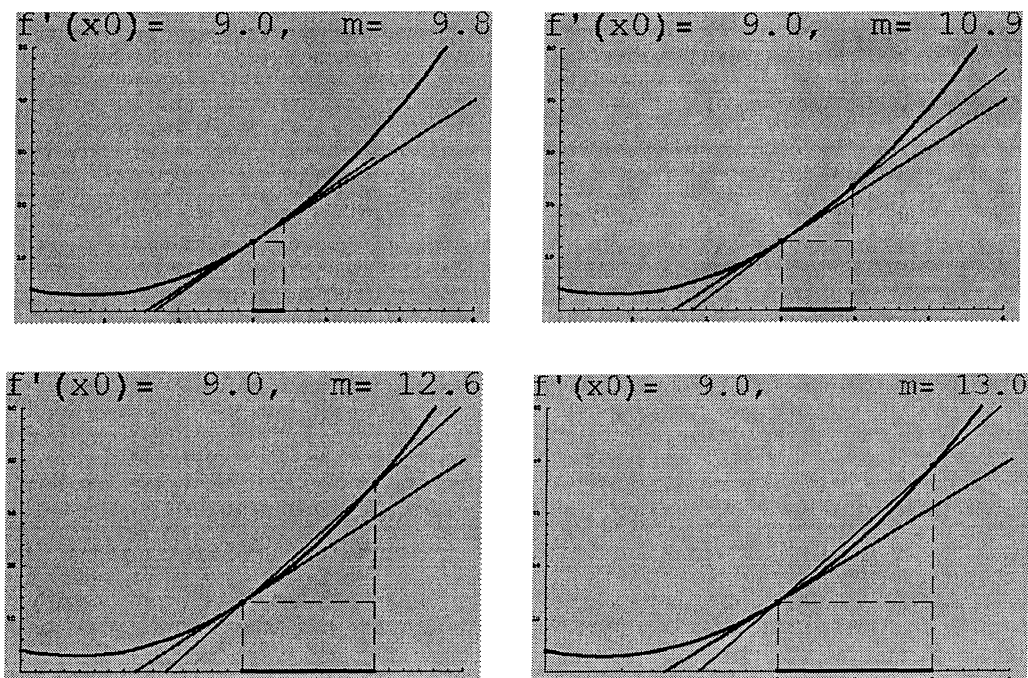
### Program 3.2

```

Clear[t]
x0 = 3; b = 5; f[x_] := 2x^2 - 3x + 4; n = 200;
t[i_] := x0 + i*(b - x0)/n
StylePrint["Secants Lines and Tangent Lines", "Text", FontSize -> 36, FontFamily -> "Times",
FontColor -> RGBColor[0, 0.4, 0], TextAlignment -> Center, FontWeight -> "Bold", Background ->
RGBColor[0.5, 1, 0.8]]
Do[Plot[{f[x], f[x0] + f'[x0](x - x0)}, {x, 0, 6}, PlotRange -> {{0, 6}, {0, 50}}, PlotStyle ->
{{Thickness[1/140], RGBColor[1, 0, 0]}, {Thickness[1/140], RGBColor[0.6, 0.6, 0.1]}},
Background -> RGBColor[0.9, 0.9, 0.7], PlotLabel -> StyleForm[StringForm["f'(x0)=`, m=`",
PaddedForm[N[f[x0]], {3, 1}], PaddedForm[N[(f[x0] - f[t[i]])/(x0 - t[i])], {3, 1}]], FontSize -> 48,
FontWeight -> "Bold", FontColor -> RGBColor[0.95314, 0.500008, 0.0429694]],
Epilog -> {{RGBColor[0, 0.4, 0.3], PointSize[1/60], Point[{x0, f[x0]}]}, {RGBColor[0, 0.4, 0.3],
PointSize[1/60], Point[{t[i], f[t[i]]]}]}, {RGBColor[0, 0, 1], Thickness[1/200],
Line[{x0 + (t[i] - x0)(-4), f[x0] + (f[t[i]] - f[x0])(-4)}, {x0 + (t[i] - x0)(4), f[x0] + (f[t[i]] - f[x0])(4)}]},
{Thickness[1/110], Line[{x0, 0}, {t[i], 0}]}, {RGBColor[0, 0, 1], Line[{t[i], f[t[i]]}, {x0, f[x0]}]},
{Dashing[{0.04, 0.02}], RGBColor[0, 0, 1], Line[{t[i], 0}, {t[i], f[t[i]]}]}, {Dashing[{0.04, 0.02}],
RGBColor[0, 0, 1], Line[{x0, 0}, {x0, f[x0]}]}, {Dashing[{0.04, 0.02}], RGBColor[0, 0, 1],
Line[{t[i], f[t[i]]}, {x0, f[x0]}]}], {i, 1, 199, 198/40}}

```

**Output:** A few frames of the animation are given below:



**Fig. 3.3. An Animation of the Secant Lines Approaching the Tangent Line**

When the animation is run in reverse, one can observe the secant lines gradually approaching the tangent line at  $x = 3$ . The top of each frame displays the derivative of the function at  $x = 3$ , and the slope  $m$  of a secant line. For full effectiveness, this animation must be presented in conjunction with the calculations in Example 2.3. ■

## 4. Conclusion

This paper described the role of two unifying elements, (a) computation and (b) visualization, of a CAS. The computational aspect (a), used by itself, will only bring limited results for the mathematics educators. If a student or an instructor is using a CAS only to perform routine calculations, then he or she is not getting the full benefit of such a powerful tool. In order to properly see the hidden meanings behind the computations, one must also resort to visualization techniques. Because of their advanced graphics and programming capabilities, modern CAS provide an ideal environment to pass back and forth between computations and visualizations. Mainly two types of visualizations are possible, static visualization and dynamic visualization, the latter being more effective. Even though not discussed in this paper, (a) computation, and (b) visualization lead into three more aspects of a CAS, namely (c) experimentation, (d) pattern-recognition, and (e) conjecture-forming. The full strength of a CAS only comes into surface, when all the unifying elements (a)-(e) are used in harmony with each other.

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