MATHEMATICS INSTRUCTIONAL TECHNOLOGY AND COLLEGE GEOMETRY

Reza Sarhangi
Mathematics Department
Towson University
8000 York Road
Towson, MD 21252, USA
rsarhangi@towson.edu

Introduction

This paper presents three computer software products useful in the introduction of some classical upper division mathematics classes such as geometry and for some related topics in abstract algebra.

Studying designs that are based on regular polygons and their constructions provides a rich source for introducing ideas in geometry and abstract algebra. This paper demonstrates a variety of ways that the *Geometer's Sketchpad* software helps the instruction of these subjects.

Perhaps the most suitable model for the exploration of hyperbolic geometry in the classroom is the *Poincaré Disk.* The two-dimensional model maps the hyperbolic plane of all space onto a unit-radius size disk. Here, hyperbolic lines are represented by arcs of circles perpendicular to the bounding circle of the unit disk. *Non-Euclid* is a Java software simulation that offers straightedge and compass constructions in both the Poincaré Disk and the Upper Half-Plane models. As an instructional tool, Non-Euclid offers instructors an opportunity to bridge the gap between developmental understanding of non-Euclidean geometry, which is based on visual explanations, and the importance of understanding theorems and concepts.

Another topic in which computers can be used is the study of rigid transformations in the plane, and the symmetry groups of one- and two-dimensional patterns. For this purpose, *Tessellation Exploration* will be presented. The Tessellation Exploration software utility is able to tessellate with 33 different types of tiles. This utility provides an environment that helps students analyze isometries used in a tiling.

Construction of Regular Polygons

Karl Fredric Gauss, a young student of nineteen, was the first to prove the impossibility of the construction of a class of regular polygons. He proved that the construction of a regular polygon having an odd number of sides is possible when, and only when, that number is either a *Fermat number*, a prime of the form $2^k + 1$, where $k = 2^n$, $n = 0,1,2,\ldots$, or is made up by multiplying together different Fermat primes [1]. Such a construction is not possible for regular polygons such as a heptagon or nonagon. Gauss first showed that a regular 17-gon is constructible, and then after a short period, solved the general problem.

The *Cyclotomic Extensions* in abstract algebra is a topic that ties together results from modern algebra and ancient geometric construction problems. In this topic, Gauss's claim can be proved in a fairly short argument using Galois theory [1]. Then what is left for students is to see some of these constructions to make sense of what they have learned.

Certain constructions are more straightforward. The construction of the hexagon, and subsequently equilateral triangle and dodecahedron, is a natural property of the circle; the radius of each circle divides the circle into six parts. Based on the construction of perpendicular lines, students can figure out how to construct a square and then subsequently an octagon and a 16-gon. However, dividing a circle into five congruent parts does not seem natural.

Figure 1(a) shows the construction of a regular pentagon using the Golden Cut, *G*, of radius *AB*, where $AB/AG = AG/GB$. First, *C*, the midpoint of the radius *AB* is found. At *B*, *BD* is drawn perpendicular to *AB* and equal in length to *CB*. The points *A* and *D* are joined and point *E* is found on *AD* such that $DE \cong BD$. From point *A*, an arc of radius *AE* is drawn to cut *AB* at *G*. The length *AG* divides the circle into 10 equal segments.

An outline of the proof can be stated as follows:

From $AB^2 + BD^2 = AD^2$, $AB = 2ED$, $BD = ED$, and $AG = AE$ we conclude that $AG = (\sqrt{5} - 1)\, ED$ and $GB = (3 - \sqrt{5})\, ED$. Therefore, we have $AB/AG = AG/GB$ (Figure 1.a).

To show the length *AG* divides the circle into 10 equal arcs we only need to show that $\angle BAK = \pi/5$, where $BK = AG$ (Figure 1.b). For this we consider two triangles ΔABK (which is an isosceles triangle) and ΔKBG. They are similar because $AB/BK = BK/GB$ and $\angle KBG = \angle KBA$. This shows $KG = BK = AG$ and therefore $\angle BAK = \angle AKG$. But from the similarity of the two triangles ΔABK and ΔKBG we know $\angle BAK = \angle BKG$. Therefore $\angle AKB = 2\angle BAK$ and since $\angle AKB = \angle ABK$ we conclude that $5\angle BAK = \pi$. □

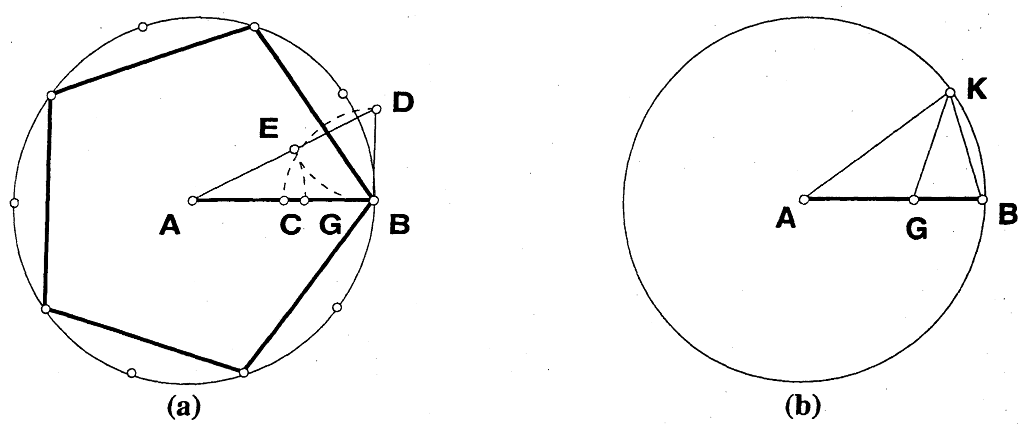

Figure 1: *The Construction of Regular Pentagon.*

The Geometer's Sketchpad Animated Construction

The Geometer's Sketchpad is a visual geometry software program, which is distributed by Key Curriculum Press. This software program is based on the rules of constructions using compass and straightedge. After becoming familiar with the Geometer's Sketchpad, it is not difficult to prepare a "script" file for construction of a pentagon and, consequently, a decagon. This animated file can be presented to the class, illustrating a step-by-step process of the construction.

To do this, we need to pull down the *File* menu and select "New Sketch" and "New Script". Press the record button, which is located on the *script* window. In the *sketch* window, select two random points *A* and *B* and construct the line segment *AB*. Then construct the midpoint *C* of *AB*. The next step is to select *AB* and point *B* and construct the line perpendicular to *AB* at *B*. Sketchpad will label this line *k*. Now use *B* as the center of a circle that passes through *C*. Select and label the point of intersection of this circle with line *k*, which will be *D*. Now construct the line segment *AD*. The next step is to make a circle, which is centered at *D* that passes through *B*. This circle will meet *AD* at *E*. Now make a circle with center at *A* that passes through *E*. This circle will intersect the line segment *AB* at *F*. Relabel *F* and call it G by double clicking the left bottom of the mouse on *F* and changing it to *G*. *G* is the Golden Cut of *AB*. Now we use *AG* to divide the circle into 10 congruent parts and construct a regular pentagon. The last step is to hide most objects keeping only the line segment *AB*, point *G*, and the pentagon. Now stop the recording. We can test the pentagon maker script by opening a new sketch and selecting two random points of *A* and *B*. Clicking on the play button, the computer shows the construction step by step.

Examples from Art and Architecture

With the same procedure explained in the above section we are able to expand our set of animated constructions to include examples from art and architecture.

Figure 2 is a Persian ceramic design, which includes carpet-like details made from solidly, colored, small, curved tiles. The division of space of the layout creates geometrical pieces. The construction of the design in this figure is based on the pentagon (and thus decagon as well). Figure 3 provides the geometrical constructions of pieces in Figure 2. Utilizing the Geometer's Sketchpad, we are able to present each piece as an animated construction, which also includes information about their measurements.

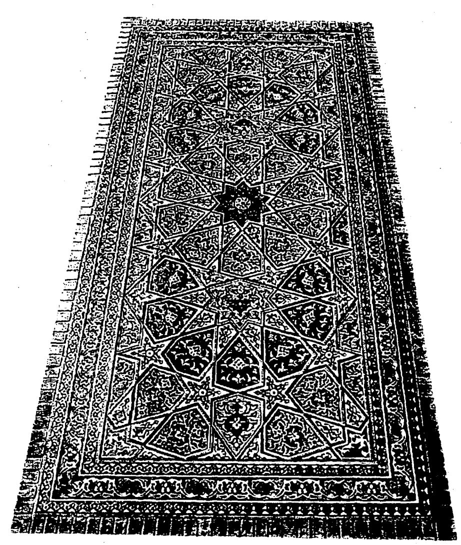

Figure 2: *A Pentagon-based Ceramic.*

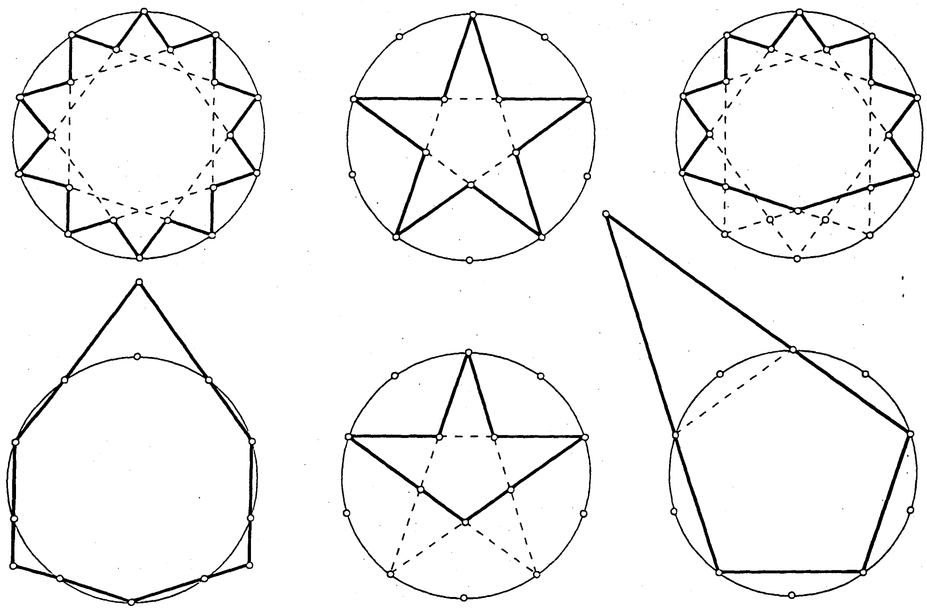

Figure 3: *Details of the parts for the underlying pattern of the ceramic in Figure2.*

Non-Euclidean Geometry and Computers

Compared to the Klein model and pseudosphere presentation of hyperbolic space, the Poincaré Disk seems more appropriate for the introduction of hyperbolic geometry in a geometry class. This model is *conformal*, which means it preserves angles. Consider that you are in the center of the disk and want to walk away toward the boundary. From your local perspective, each step you take is the same size. But from the point of view of an observer outside of this disk your steps get progressively smaller by the ratio of $(1-r^2)/2$, where r is your distance from the origin in this model.

Non-Euclid is a Java software simulation that performs geometrical constructions in both the Poincaré Disk and the Upper Half-Plane models. This utility can be downloaded from http://cs.unm.edu/~joel/NonEuclid/.

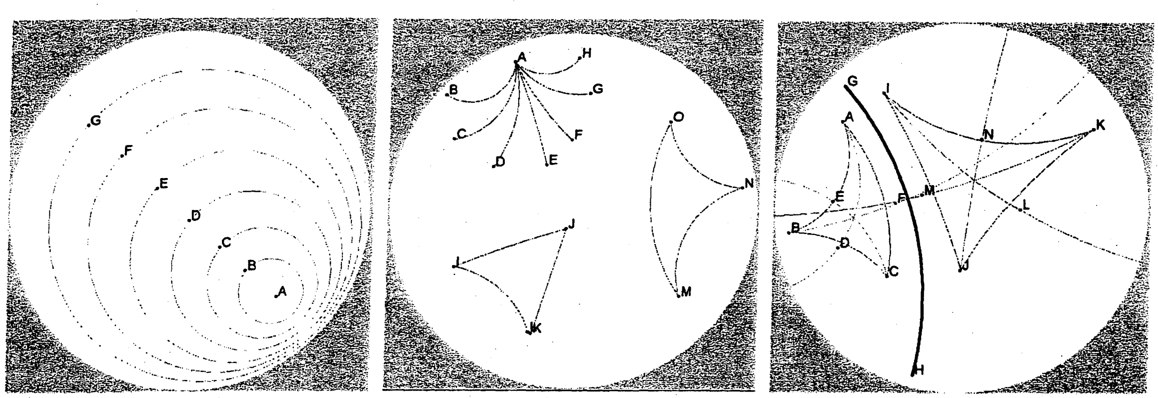

Figure 4: **(a)** *Concentric circles,* **(b)** *Line segments that share an endpoint and two triangles,* **(c)** *TriangleΔ ABC and its interior angle bisectors (Three angle bisectors are also concurrent in the hyperbolic geometry) and the reflection of this triangle under the line GH.*

Tessellation Exploration and Wallpaper Patterns

Tessellation Exploration is a software utility which is distributed by Tom Snyder Productions and designed to create tilings using basic geometric figures such as triangles and quadrilaterals. The software is able to tessellate with 33 different types of tiles. The software has been developed based on Heech's classification [2] of the 28 types of asymmetric tiles that can fill the plane in an *isohedral* manner without using reflections. The other five tiles in this software utilize reflections. An isohedral tiling is defined as selecting two congruent figures such that there always exists a symmetry motion that will move one of the figures exactly onto the other.

Polya illustrated the 17 wallpaper patterns in his article "Über die Analogie der Kristallsymmetrie in der Ebene" published in Zeitschrift für Kristallographie in 1924 [3]. The following figures are renditions of two of Polya's illustrations that the author of this article has created with the help of Tessellation Exploration. Figure 5(a) has been identified as D°_4 in Polya's paper. The basic shape to construct this pattern in Tessellation Exploration is a triangle. The isometries employed are a reflection and a quarter rotation. The mathematical notation for this pattern is *P4m.* It belongs to the square lattice of wallpaper patterns and its highest order of rotation is 4. It can be generated by 1/8 of its square unit. The other figure is the rendition of D°_3 Polya's illustration, which has been created, based on a triangle and two isometries of a reflection and a rotation of 120°. The mathematical notation for this pattern is *P31m.* It is in the hexagonal lattice with the highest order of rotation 3. It can be generated by 1/6 of a hexagon unit.

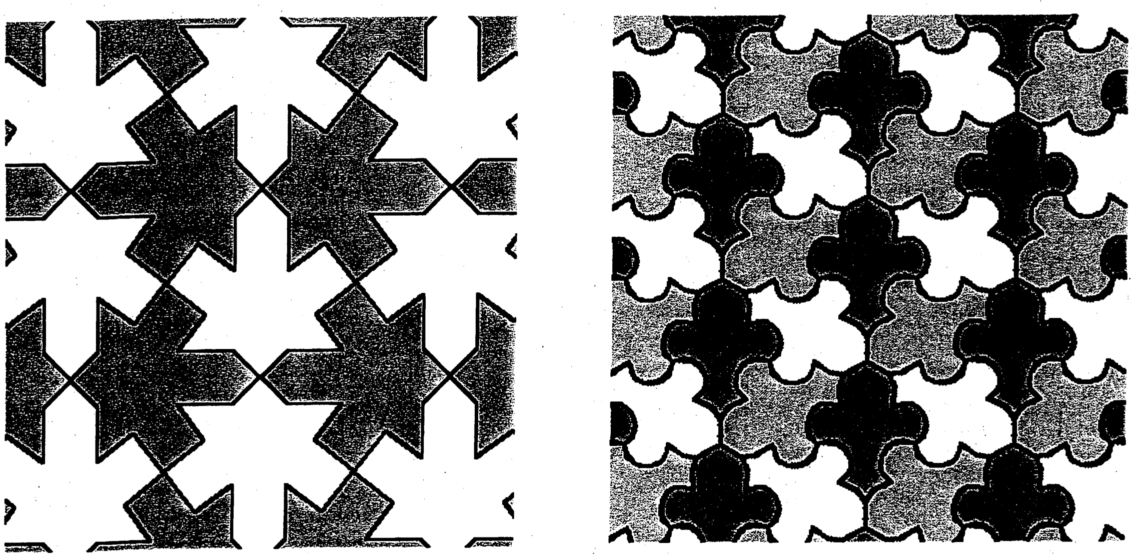

Figure 5: **(a)** *A P4m Wallpaper Pattern, and* **(b)** *a P31m Wallpaper Pattern.*

References:

[1] J. A. Gallian, *Contemporary Abstract Algebra*, 4th edition, Houghton Mifflin Company, 1998.
[2] B. D. Martin, R. Sarhangi, *Symmetry, Chemistry, and Escher's Tiles*, 1998 Bridges Proceedings, Gilliland Printing, Kansas, 1998.
[3] D. Schattschneider, *M. C. Escher, Visions of Symmetry*, W. H. Freeman and Company, New York, 1990.