USING MAPLE®'S STUDENT PACKAGE TO EXPLORE DEFINITE INTEGRALS AND NUMERICAL INTEGRATION

William H. Richardson, Jr., Ph.D.
Francis Marion University
Department of Mathematics
PO Box 100547
Florence, SC 29501
wrichard@fmarion.edu

Teaching a course in integral calculus requires graphic images that depict areas under curves. Since these areas may be estimated by summing areas of appropriately chosen rectangles or trapezoids, images of these figures together with the curve are needed as well. Using MAPLE® the instructor can produce such images quickly and easily. Simple MAPLE® procedures can be written to produce specialized images for graphically illustrating concepts related to the trapezoidal rule and Simpson's rule.

The MAPLE® kernel is supplemented by many software *packages* that offer the user a collection of procedures responding to similar needs. One such package is the *student* package whose procedures offer ways to facilitate teaching mathematics. Among procedures in the student package are *leftbox*, *middlebox*, and *rightbox*, which help illustrate the method used to approximate areas under a curve. Procedures *leftsum*, middlesum, and *rightsum* compute the associated sum of areas. Consider that the definite integral given by $\int_a^b f(x)\,dx$ can be introduced as

$$\int_a^b f(x)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \qquad (1)$$

where Δ represents a partition of $[a,b]$ and c_i is any point in the ith subinterval. Frequently c_i is taken as the left hand or right hand endpoint of the ith subinterval where the subintervals are of equal length. For a given function, the MAPLE® *leftbox*, *middlebox*, and *rightbox* procedures (commands) create effective and colorful depictions of n rectangles the sum of whose areas approximates the definite integral over $[a,b]$. The MAPLE® user simply provides a value for n, the number of desired subintervals, along with the function to be integrated. For example, consider the area bounded by the graph of $f(x) = 8 - 0.5x^2$, the x-axis, the line $x = 1$, and the line $x = 3$. With $f(x) = 8 - 0.5x^2$ previously defined and ten subintervals required, the following MAPLE® command

```
rightbox(f(x),x=1..3,10,'shading'=grey,color=blue);
```
(2)

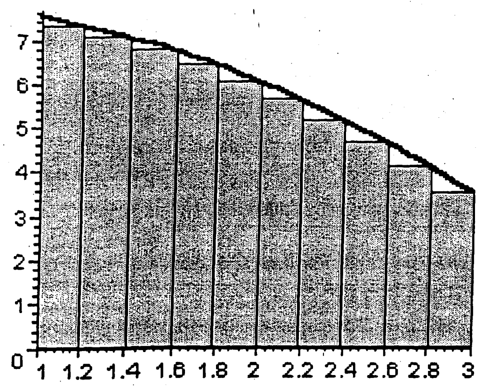


Figure 1. Command *rightbox* is used to depict rectangles whose sum approximates area under the curve.

results in the image shown in Figure 1. Similar graphics are obtained using the commands *middlebox* and *leftbox*, for which c_i in (1) is taken to be the midpoint and the left hand endpoint of any subinterval, respectively.

As a follow-up, the *student* package routine *rightsum* (*middlesum*, *leftsum*) can be used to compute the associated sum of areas. This is especially useful after evaluating a few sums on the chalkboard, a process that can consume much class time. The *rightsum* command shown in (3) was used to produce the result (4). Note that the MAPLE® command *rightsum* implements the sum exactly as it would appear on paper since, for $a = 1$, $b = 3$, and $n = 10$, $x_i = a + \left(\frac{b-a}{n}\right)i = 1 + \frac{i}{5}$ for $i = 1, \ldots, 10$ indicates the right-hand endpoint of the ith subinterval.

```
DesiredArea:=rightsum(f(x),x=1..3,10)=
                         value(rightsum(f(x),x=1..3,10));
```
(3)

$$DesiredArea := \frac{1}{5}\left(\sum_{i=1}^{10}\left(8 - 0.5\left(1 + \frac{i}{5}\right)^2\right)\right) = 11.26000000 \tag{4}$$

More examples of this type can be explored quickly using MAPLE®. It is easy to compute iteratively the sum of areas for increasing numbers of subintervals to demonstrate the convergence of the sum to the area represented by the definite integral. Also, the inert commands *Sum* and *Limit* may be used to display symbolically the limit in (1) as well as compute that limit as the number of subintervals increases without bound.

To further investigate the method of approximating $\int_1^3 8 - 0.5x^2\,dx$, the following commands shown in (5) determine lower and upper approximations for the relevant area under the graph of $f(x) = 8 - 0.5x^2$ for increasing numbers of equally spaced subintervals. An abbreviated result is shown in (6).

```
> for n from 10 by 20 to 520 do
>     areaLo:=value(rightsum(f(x),x=1..3,n)):
>     areaHi:=value(leftsum(f(x),x=1..3,n)):
>     printf("\n               For n = %3d:   %8.4f <  DesiredArea  <
%8.4f",n,areaLo,areaHi):
> od:
```
(5)

```
For n =  10:    11.2600 <  DesiredArea  <  12.0600
For n =  30:    11.5326 <  DesiredArea  <  11.7993
For n =  50:    11.5864 <  DesiredArea  <  11.7464
                                ⋮                          (6)
For n = 470:    11.6582 <  DesiredArea  <  11.6752
For n = 490:    11.6585 <  DesiredArea  <  11.6748
For n = 510:    11.6588 <  DesiredArea  <  11.6745
```

MAPLE® offers inert commands that are identified by capitalizing the first character of a command. Their purpose is to express a computation without actually computing a numerical result. For example, the command

```
Int(f(x),x=1..3);
```
(7)

results in

$$\int_1^3 8 - 0.5\,x^2\,dx \qquad (8)$$

whereas

```
int(f(x),x=1..3);
```
(9)

results in 11.6666667 . Using the commands *limit* and *sum* together with their inert forms, the MAPLE® statement

```
Limit(Sum('(8.0-0.5*(1.0+i*2.0/n)^2)*(2/n)','i'=1..n),n=infinity)=
     limit(sum('(8-0.5*(1.0+i*2/n)^2)*(2/n)','i'=1..n),n=infinity);
```
(10)

produces

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2\left(8.0-0.5\left(1.0+\frac{2.0\,i}{n}\right)^2\right)}{n}\right)=11.66666667 \qquad (11)$$

In this case, c_i in (1) is taken to be the right-hand endpoint of any subinterval. Classroom demonstrations typically include chalkboard evaluations of limits like the one shown above. These demonstrations can be supplemented with various limit evaluations corresponding to other choices for c_i, and hence other Riemann sums.

Numerical integration becomes necessary when the integrand has no antiderivative or the function to be integrated is represented by a collection of data pairs taken by experiment. Graphs that help illustrate the midpoint, trapezoidal, and Simpson's rules are not easily rendered on the chalkboard. While the MAPLE® command *middlebox* can be used to illustrate the midpoint rule, no intrinsic command is offered to help students visualize the approximations implemented when using the trapezoidal rule or Simpson's rule. A procedure was written for each so that these rules could be better illustrated. Called *myTrap*, the first of these was invoked using

```
myTrap(f,0.5,1.5,4);
```
(12)

where f represents the previously defined function $f(x) = -x^4 + 2x^3 - 2x^2 + 2x + 0.5$, parameters 0.5 and 1.5 bound the illustration on the x-axis, and the parameter 4 indicates that four equally-space intervals are to be used. The result is shown in Figure 2. The MAPLE® student package includes the procedure *trapezoid* that uses the trapezoidal rule to approximate values for definite integrals. For example, the *trapezoid* command returns the value 1.294921875 as the approximate value for the shaded area in Figure 2 when 4 equally-space intervals are used. This deviates from the true value of the area, $\frac{317}{240}$, by an amount approximately equal to 0.025911458.

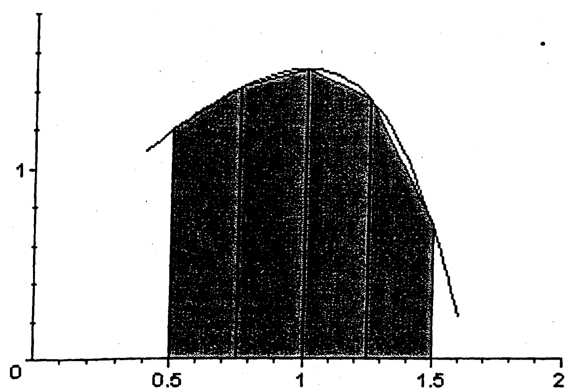


Figure 2. Procedure *myTrap* illustrates trapezoidal approximations for $f(x) = -x^4 + 2x^3 - 2x^2 + 2x + 0.5$

A second procedure called *mySimp* was written to allow the instructor to graphically illustrate how Simpson's rule approximates the definite integral corresponding to an area under a curve. Figure 3 shows the image produced by a call to *mySimp* with

```
mySimp(f,0,2,3,12,0);
```
(13)

In this case, $f(x) = 1.5 + \cos 4.1x$, the definite integral is approximated from $x = 0$ to $x = 2$ using $3*2 = 6$ equally spaced subintervals (3 pairs of adjacent subintervals). The value 12 allows the user to control the shading process while the last parameter 0 provides user control over the lower horizontal edge of the graphing window. Note the 3 parabolas – one for each adjacent pair of subintervals. Any parabola intersects the graph of $f(x)$ at the three points whose x-coordinates are the left-hand endpoint, the midpoint, and the right-hand endpoint of its pair of adjacent

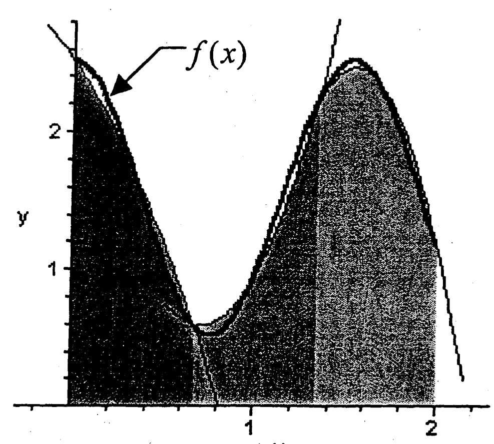


Figure 3. Procedure *mySimp* illustrates Simpson's Rule approximations for $f(x) = 1.5 + \cos 4.1x$

subintervals. For this case, when 6 equally spaced intervals are used, the intrinsic MAPLE® command *simpson* returns the value 3.235121790 as the approximate value for the shaded area in Figure 3. The following MAPLE® command

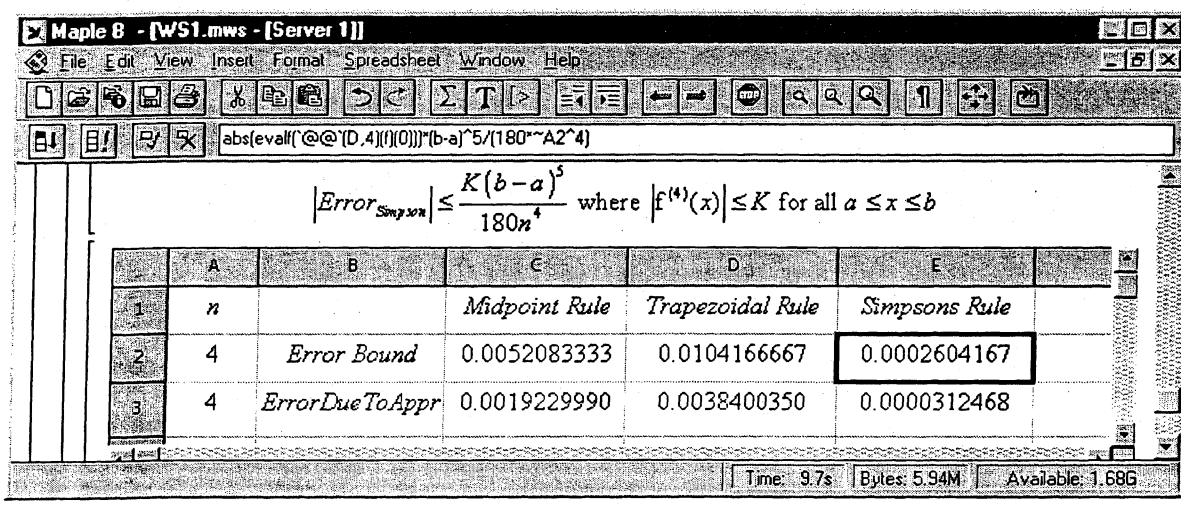
```
Int(f(x),x=0..2)=int(f(x),x=0..2);
```
(14)

returns

$$\int_0^2 \cos(4.1\,x) + 1.5\,dx = 3.229446477 \qquad \textbf{(15)}$$

indicating an error due to the approximation using Simpson's Rule (with $n = 6$) of around 0.005675313.

As a final example of the versatility of MAPLE® consider the partial MAPLE® worksheet shown in Figure 4. It contains an Excel®-like spreadsheet with theoretical bounds on errors due to approximating the definite integral of the function $f(x) = e^{-x^2}$ from $x = 0$ to $x = 1$. The midpoint rule, the trapezoidal rule, and Simpson's rule were used with 4 subintervals. The last row of data provides the actual errors seen using the 3 methods in this case with $f(x) = e^{-x^2}$. Note that MAPLE® provides for the use of equation editing within text contained in the worksheet (above the spreadsheet). Also notice that, similar to Excel®, a cell may contain a formula that references other cells in the same spreadsheet. The highlighted cell (E2) references cell A2, utilizes a fourth derivative, and references other previously defined values from the worksheet. This can be seen in the text box just below the main tool bar.



	A	B	C	D	E
1	*n*		*Midpoint Rule*	*Trapezoidal Rule*	*Simpsons Rule*
2	4	*Error Bound*	0.0052083333	0.0104166667	0.0002604167
3	4	*ErrorDueToAppr*	0.0019229990	0.0038400350	0.0000312468

Figure 4. Part of a worksheet showing a MAPLE® spreadsheet that contains actual errors and theoretical error bounds related to using the midpoint, trapezoidal, and Simpson's rules to approximate $\int_0^1 e^{-x^2}\,dx$.