IF NEWTON HAD A SPREADSHEET WOULD HE HAVE DEVELOPED CALCULUS?

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Abstract
Newton's work on dynamics is described in his three books of *The Principia.* This is a work of major importance since it develops the fundamental ideas with which we may describe the mathematics of motion. Newton's arguments are based on geometric ideas and, in particular, for the motion of the planets around the Sun he uses arcane geometrical properties of the ellipse to show that the planets do indeed move in elliptical orbits. Although he didn't develop the calculus for the motion of the planets, the underlying ideas are there. He uses the concept of a limit, in a manner which we would recognise as a derivative, to obtain planetary orbits. We shall follow Newton's approach and show how to implement his method as a set of recurrence relations on a spreadsheet. Our spreadsheet implementation will be used to calculate the parameters of the orbit of the Earth.

Motion of the planets around the Sun

A very good account of Newton's work, in particular his work on planetary motion, is given by Thrower (1990). In this section we use the spreadsheet, following in Newton's footsteps (Cajori 1934, Stein 1996) to develop the elliptic orbits of the planets around the Sun. Newton's approach was developed using arcane geometric properties of the ellipse, which were well-known in Newton's time. We shall follow the main thrust of his argument, introducing our own notation to facilitate the development of suitable recurrence relations.

Suppose that the motion is described as a set of discrete equal time-steps, Δt, and consider the motion in the time interval $t_r \leq t \leq t_r + \Delta t = t_{r+1}$. The notation is described in Figure 1.

The Sun, *S*, is fixed at the origin and the planet, *P*, has position $P_r, (x_r, y_r)$ at time t_r. At this time the planet has a velocity v_r along the tangent to the trajectory so that, in the absence of any other force, Newton's first law implies that the planet would move to the point P'_{r+1} with $P_r P'_{r+1} = v_r \Delta t$.

However, the effect of the Sun is to produce a gravitational force which obeys the inverse square law. Following Newton, we shall assume that this effect is the same as that of a suitable impulse applied to the planet at P'_{r+1}. This impulse will give the planet a velocity u_r which we assume to be in a direction parallel to the vector $\overrightarrow{P_r S}$ so that the planet would move to the point P_{r+1} with $P'_{r+1} P_{r+1} = u_r \Delta t$.

The reason for choosing the direction parallel to $\overrightarrow{P_r S}$ is to ensure that the motion obeys Keppler's second law as follows:

$$\Delta SP_{r-1}P = \Delta SP_r P'_{r+1}$$
$$= \Delta SP_r P_{r+1}$$

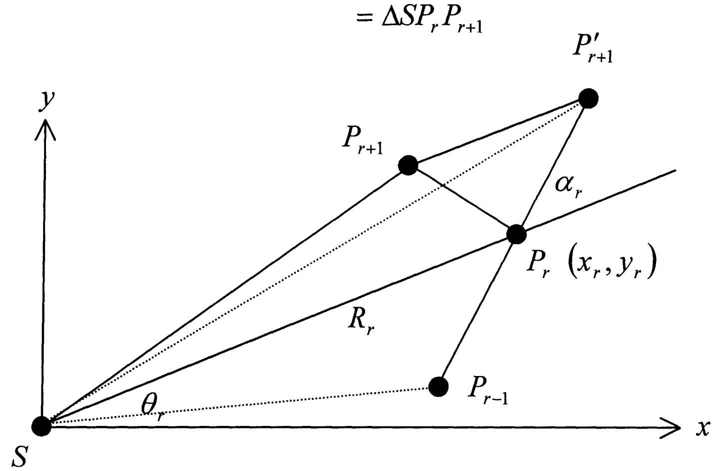

Figure 1 Notation for motion of planet around the Sun.

Hence the radius vector sweeps out equal areas in equal times.
We can see that at P_{r+1} the planet has a total velocity, V_r, given by
$V_r^2 = v_r^2 + u_r^2 - 2v_r u_r \cos\alpha_r$

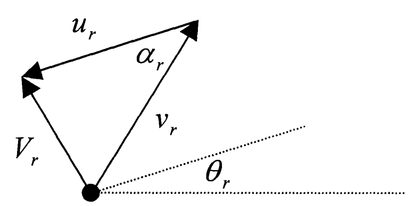

Figure 2 Total velocity at P_{r+1}.

The force, F_r, at P_r is given by

$$F_r = \frac{GMm}{R_r^2},$$

where G is the universal gravitational constant and M and m are the masses of the Sun and the planet respectively. The force is directed towards the Sun in the direction $\overrightarrow{P_r S}$.

The impulse is given by

$$J_r = F_r \Delta t,$$

and, since $u_r = \frac{J_r}{m}$, it follows that

$$u_r = \frac{GM}{R_r^2}\Delta t.$$

We are now in the position to set up the recurrence relations:

$$R_r^2 = x_r^2 + y_r^2 \tag{1}$$

$$u_r = \frac{GM}{R_r^2}\Delta t \tag{2}$$

$$\theta_r = \arctan\left(\frac{y_r}{x_r}\right) \tag{3}$$

$$\alpha_r = \arccos\left(\frac{R_r^2 + (v_r \Delta t)^2 - R_{r-1}^2}{2R_r v_r \Delta t}\right) \tag{4}$$

$$x_{r+1} = x_r + \Delta t(v_r \cos(\theta_r + \alpha_r) - u_r \cos\theta_r) \tag{5}$$

$$y_{r+1} = y_r + \Delta t(v_r \sin(\theta_r + \alpha_r) - u_r \sin\theta_r) \tag{6}$$

$$v_{r+1}^2 = V_r^2 = v_r^2 + u_r^2 - 2v_r u_r \cos\alpha_r \tag{7}$$

We notice that equations (1), (3) and (4) are purely geometric relations. Equation (7) is a kinematic relation, easily derived from Figure 2. Equation (2) describes the velocity of the planet due to the impulse at time t_r. Equations (5) and (6) describe the motion of the planet in the time interval (t_r, t_{r+1}). We note here that the recurrence relations (5) and (6) are those that are obtained by using Euler's method to approximate the differential equations $\dot{x} = V_x$, $\dot{y} = V_y$.

These recurrence relations are solved subject to suitable initial conditions and we shall consider the motion of the Earth in its orbit around the Sun with the following data (Allen 1976):

$v_0 = 3.029 \times 10^4 ms^{-1}$, $R_0 = 1.471 \times 10^{11} m$ (perihelion values),

$M = 1.989 \times 10^{30} kg$, $m = 5.974 \times 10^{24} kg$, $G = 6.673 \times 10^{-11} Nm^2 kg^{-2}$.

In Figure 3 we show the spreadsheet with a time-step of one day $(86400s)$ and we develop the solution in the first ten time-steps. In Figure 4 we show the trajectory in the first quadrant which takes approximately 90 days.

	A	B	C	D	E	F	G	H	I	J
1	Motion of Earth around Sun									
2										
3		Δt		G	Ro	M	m	Vo		
4		86400		6.673E-11	1.471E+11	1.989E+30	5.974E+24	30290		
5										
6	r	t_r	x_r	y_r	R_r	theta_r	alpha_r	v_r	u_r	ΔA_r
7	0	0	1.471E+11	0	1.471E+11	0	1.570796327	30290	529.9614809	
8	1	86400	1.471E+11	2617056000	1.47123E+11	0.017789123	1.553007204	30290	529.7937908	1.92484E+20
9	2	172800	1.47054E+11	5233297760	1.47147E+11	0.035572524	1.552715457	30285.20879	529.6206566	1.92484E+20
10	3	259200	1.46963E+11	7847912091	1.47172E+11	0.053350021	1.552426711	30280.26385	529.4421438	1.92484E+20
11	4	345600	1.46826E+11	10460087137	1.47198E+11	0.071121436	1.552141056	30275.16689	529.2583199	1.92484E+20
12	5	432000	1.46643E+11	13069012669	1.47224E+11	0.08888659	1.551858582	30269.91969	529.0692539	1.92484E+20
13	6	518400	1.46414E+11	15673880365	1.47251E+11	0.10664531	1.551579376	30264.52407	528.8750171	1.92484E+20
14	7	604800	1.46141E+11	18273884098	1.47279E+11	0.124397423	1.551303526	30258.98189	528.6756822	1.92484E+20
15	8	691200	1.45822E+11	20868220214	1.47307E+11	0.142142759	1.551031117	30253.29507	528.4713238	1.92484E+20
16	9	777600	1.45457E+11	23456087814	1.47336E+11	0.15988115	1.550762234	30247.46557	528.2620181	1.92484E+20
17	10	864000	1.45048E+11	26036689034	1.47366E+11	0.177612432	1.55049696	30241.4954	528.0478431	1.92484E+20

Figure 3 Spreadsheet implementation of recurrence relations (1)…(7).

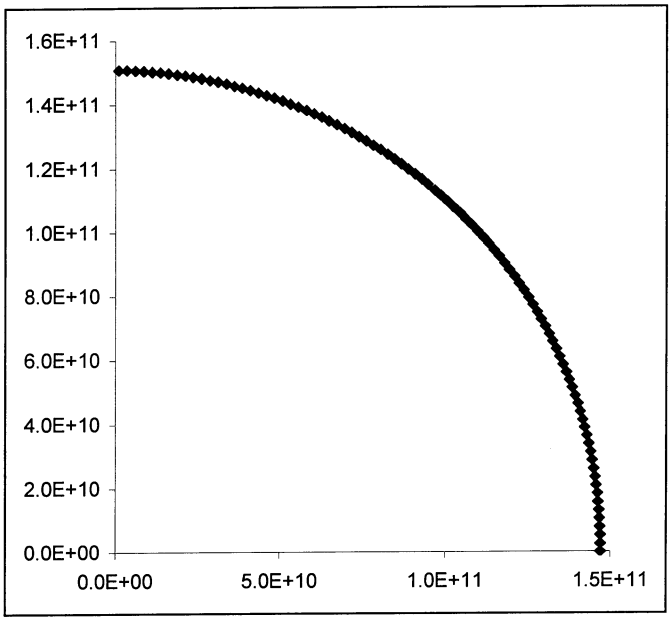

Figure 4 Trajectory of Earth in its motion around the Sun.

We see, qualitatively, that the trajectory closely resembles an ellipse.

The angular momentum is given by

$$L = mR\frac{d\theta}{dt}.$$

Now, since the area swept out and the angular momentum are related by

$$\frac{dA}{dt} = \frac{L}{2m},$$

Kepler's second law is equivalent to the conservation of angular momentum.

	K	L	M	N
3	L		Eo	
4	2.662E+40		-2.650E+33	
5				
6	**L_r**	**ΔL_r**	**E**	**ΔE_r**
7	2.66181E+40			
8	2.66237E+40	0.013987	-2.6497E+33	0
9	2.66239E+40	0.014502	-2.6489E+33	0.032187
10	2.66240E+40	0.015012	-2.6489E+33	0.032706
11	2.66241E+40	0.015516	-2.6488E+33	0.03322
12	2.66243E+40	0.016015	-2.6488E+33	0.033728
13	2.66244E+40	0.016506	-2.6488E+33	0.034229
14	2.66245E+40	0.016992	-2.6488E+33	0.034723
15	2.66247E+40	0.017471	-2.6488E+33	0.03521
16	2.66248E+40	0.017944	-2.6488E+33	0.035691
17	2.66249E+40	0.018409	-2.6488E+33	0.036165

Figure 5 Angular momentum and energy at each time-step.

Also, the system moves in such a fashion that energy is conserved, *i.e.*

$$E = \frac{1}{2}mV^2 - \frac{GMm}{R} = \text{constant}.$$

The angular momentum, L_r, and the energy, E_r, together with the percentage changes, at each time-step are shown in Figure 5.

We see that the errors in L_r and E_r are small; in fact the average errors are of the order 0.02% and 0.04% respectively which are of the same order as the errors in the data.

Conclusions

Newton developed the proof that planets orbit the Sun in ellipses using arcane geometrical properties of the ellipse. He based the proof on his law of gravitation and on Kepler's second law. Newton's approach allows us to develop a set of recurrence relations which may be implemented on a spreadsheet. Solving these equations with data for the Earth produces a trajectory which qualitatively has the appearance of an ellipse. When we use the data to estimate the parameters such as the major and minor axes, the semi-latus rectum and the eccentricity we find that the errors in the predicted are within the accuracy off the given data.

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