Integrating the Cassiopeia into the Mathematics Curriculum

Lucy Kabza
Randy Wills
Department of Mathematics
Southeastern Louisiana University
Hammond, LA 70402
lkabza@selu.edu
rwills@selu.edu

For more than a decade there has been a national move towards the effective use of technology in the mathematics curriculum. The most common technologies used in the mathematics curriculum are graphing calculators and PCs running mathematical software such as Mathematica and Maple. The advantage of the graphing calculator is that it is portable so that the student always has access to the technology. The disadvantage of the graphing calculator is that it is limited in what it can do and has a long learning curve. The advantage of using a PC running mathematical software is that one can do far more complex problems than would be possible on a calculator. Also, software such as Mathematica and Maple are actually programmable computer algebra systems capable of doing virtually any problem in the undergraduate curriculum. Also, these programs are invaluable tools for professionals in both industry and academia. Exposure to these tools is something that makes a student more marketable. The problem with these tools is that the syntax is difficult to learn and the programs themselves are very expensive. We found a way to exploit the advantages of both the portability of the graphing calculator and the power of the symbolic manipulation package Maple. The Cassiopeia A-21S is a hand-held computer extender that can run a menu driven version of Maple. The A-21S is somewhat larger than a calculator, and it runs under the Windows CE operating system, which has point-and-click technology.

So far the Cassiopeias have been used in: Calculus I, II, Linear Algebra, Statistics, Geometry. The following are the advantages of using Cassiopeias over other technologies: small and portable, extremely easy to use, runs MAPLE (menu driven), no need to redesign the course, no need to use the computer lab.

Cassiopeia is most useful when the objective is not the skill (e. g. differentiation, integration), but understanding the problem and setting up a function, an equation, or an integral. The rest of the calculations can be executed by Cassiopeias. The following are examples of types of problems that are suitable for Cassiopeias: optimization problems (differentiation), related rates (implicit differentiation, solving equations symbolically), graphing functions (differentiation and solving equations to find critical numbers), volume, arc length, surface area, Riemann sums, power series, graphs of Taylor polynomials, in Linear Algebra from matrix operations to eigenvalues, and graphing implicit eqations in Taxicab Geometry or any other geometry.

In many of the above problems the calculations are not difficult, but very tedious. With Cassiopeias the applied problems, that all books abound in, can now be explored since the calculations do not stand in the way. All the examples can be just as well investigated using MAPLE software on a PC, but the students would need to master some of the syntax and some of the commands, which from our experience proved to be difficult. One also loses the instantaneous access to the software unless the class is held in the computer laboratory. As you can see, not having Cassiopeias adds a number of obstacles to teaching using advanced technologies.

The following are explicit examples that proved to be very user-friendly when using Cassiopeias, and vary hard to do, if not impossible, without it. Students can also answer a variety of different questions that cannot be answered otherwise.

Calculus:

Example (Calculus - Larson/Hostetler/Edwards) :

Approximating the Area of a Plane Region between the given curve and the x-axis over the given interval:

$y = x^2 - x^3$ [-1, 0]

n -subintervals of length $\Delta x = \frac{1}{n}$.

$A_n^r = \sum_{i=1}^{n} f(-1 + i \cdot \Delta x) \cdot \Delta x$ - right endpoints used to find the height,

$A_n^l = \sum_{i=0}^{n-1} f(-1 + i \cdot \Delta x) \cdot \Delta x$ - left endpoints used to find the height.

After simplifying on Cassiopeias we obtain:

$$A_n^r = \frac{-12n + 5 + 7n^2}{12n^2} \quad \text{and} \quad A_n^l = \frac{7n^2 + 12n + 5}{12n^2}$$

The area can be approximated by evaluating the above expressions for various values of n, and the exact area obtained by taking the limit as $n \to \infty$, which can be done by hand or on the Cassiopeia. Students can easily make the observation that $\lim_{n \to \infty} A_n^r = \lim_{n \to \infty} A_n^l = A$.

Linear Algebra:

Cassiopeias can be used throughout the course, since at all times some kind of manipulation of matrices is being done. But in particular, it is very useful in the second linear algebra course, when not the skills are emphasized, but the ability to translate problems into equations, and hence various algebraic operations on matrices have to be performed, and various quantities have to be computed, e. g., powers of matrices, eigenvalues, determinants, inverses, Gauss-Jordan elimination, etc.

These computations are needed when studying numerous applications, such as: constructing curves and surfaces through specified points - determinants , Markov Chains - row reduction, graph theory - powers of matrices, and computer graphing - matrix multiplication.

Eigenvalues and Eigenvectors.
Objective: solving the equation $A\vec{x} = \lambda\vec{x}$.
Example (Markov Chain):
This problem deals with leasing automobiles. The objective is to predict the distribution of future leases. Or in other words to find the steady-state vector if there is one.
Available vehicles: sedan, sports car, minivan, sport utility.
Transition matrix:

Current Lease				Next Lease
Sedan	Sp. Car	Mvan	SUV	
0.80	0.10	0.05	0.05	Sedan
0.10	0.80	0.05	0.05	Sp. Car
0.05	0.05	0.80	0.10	Minivan
0.05	0.05	0.10	0.80	SUV

This table is based on previous records. The initial distribution is $\vec{x}_0 = \begin{pmatrix}0.40 & 0.20 & 0.20 & 0.20\end{pmatrix}^T$, and the distribution a year later is computed by finding the product of the matrices $\vec{x}_1 = A\vec{x}_0$. The number of future leases is predicted by $\vec{x}_{n+1} = A\vec{x}_n = A^n\vec{x}_0$. The long-range behavior of the process is determined by the eigenvalues and eigenvectors of the transition matrix A. We find them using the Cassiopeia: since there is no command EIGENVALUES in the pull down menu, we need to find the roots of the characteristic polynomial obtained from equation $\det(A-\lambda I) = 0$. The PC MAPLE does compute the eigenvalues directly, but we find it more stimulating to find them the "long way". The eigenvalues are: $\lambda_1 = 1$, $\lambda_2 = 0.8$, $\lambda_3 = \lambda_4 = 0.7$. To determine the eigenvectors we use the Gauss-Jordan Elimination on $A - \lambda I$ (we solve $(A-\lambda I)\vec{x} = 0$ for each λ). We obtain Y and its inverse Y^{-1}. The steady state vector has to be computed by hand, since MAPLE does not compute the limits of the expressions that involve matrices:

$$\vec{x}_n = Y\begin{pmatrix}1 & & & \\ & 0.8 & & \\ & & 0.7 & \\ & & & 0.7\end{pmatrix}^n Y^{-1}\vec{x}_0$$

It is easy to see that when $n \to \infty$ the only eigenvalue that matters is the first one, so $DY^{-1}\vec{x}_0$ will result in just one nonzero entry, the first entry in $Y^{-1}\vec{x}_0$:

$$\begin{bmatrix}0.25\\0\\0\\0\end{bmatrix},$$

and multiplying this by matrix Y results in a steady state vector.

Euclidean and Taxicab Geometries

Now we are going to present some problems in geometry that proved to be very interesting and accessible to students of all levels.

Let $A(x_1, y_1)$ and $B(x_1, y_2)$ be points in the lane. The Euclidean distance between the points A and B is given by

$$d_E(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} .$$

The Taxicab distance between the points A and B is given by

$$d_T(A,B) = |x_2 - x_1| + |y_2 - y_1| .$$

The difference between the two definitions of distance is easily shown on the following picture:

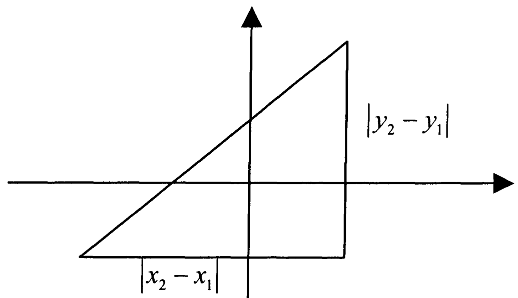

Figure 1. Distance between points in Euclidean and Taxicab Geometries.

It is clear that $d_E(A,B) \le d_T(A,B)$ with equality occurring when AB is a horizontal or vertical line segment. The difference between the two geometries is very apparent when one takes a look at the shapes of various geometric figures.

Circles

The equations of the circle in Euclidean and Taxicab Geometry are respectively

$$(x-h)^2 + (y-k)^2 = r^2 \text{ and } |x-h| + |y-k| = r .$$

Their graphs are (with center at the origin):

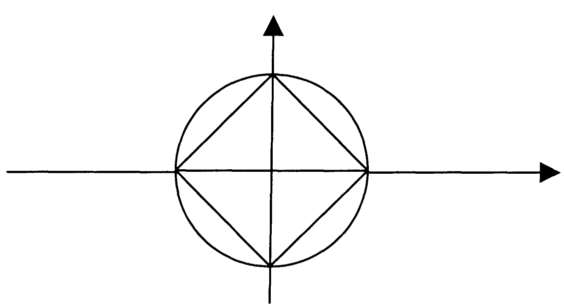

Figure 2: Circles in Euclidean and Taxicab Geometries.

To graph the above equations we take advantage of the MAPLE capabilities of graphing curves defined implicitly.

Easy as 1-2-3

In Euclidean Geometry, the lengths of the sides of a triangle determine the triangle uniquely. There is also no triangle with sides of length 1, 2, 3 respectively.

Problem: Does there exist a triangle in Taxicab Geometry where the sides have lengths 1, 2, 3? If so, is the triangle unique?

Solution: Let $A(0,0)$ and $B(1,0)$ be two points in the plane. Clearly the length of $\overline{AB}$ is 1. For a 1-2-3 triangle to exist we must find a point $C(x,y)$ such that $d_T(A,C)=3$ and $d_T(B,C)=2$. To determine if C exists we construct taxicab circles centered at A and B of radii 3 and 2 respectively. If the circles intersect, then the 1-2-3 triangle exists in Taxicab Geometry. The equations of these circles are:

$$|x|+|y|=3 \text{ and } |x-1|+|y|=2.$$

The graphs of these circles are shown below.

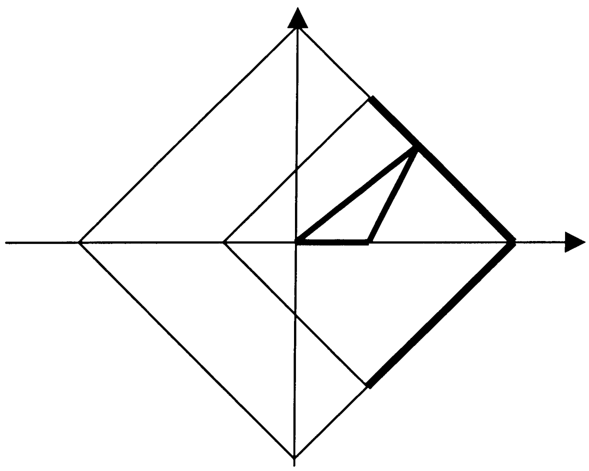

Figure 3. 1-2-3 triangle in Taxicab Geometry.

We see that an infinite number of 1-2-3 triangles exist in taxicab geometry.

Problems for further study in Taxicab Geometry:
1. Find perpendicular bisectors.
2. Find equilateral triangles.
3. Try to find an analog of the Pythagorean Theorem in Taxicab Geometry.
4. Analyze the various conic sections using taxicab geometry.

In conclusion, we would like to say that teaching with Cassiopeias has been very rewarding. This is one technology that both the students and the instructor do not have to spend time learning. This makes it possible to concentrate on mathematics, since that is the prime objective of all mathematics courses. Both authors have experience in using Maple for teaching and research on a PC and found that the Cassiopeias greatly enhanced student learning in the courses.

1. Steven J. Leon, *Linear Algebra with Applications,* 5th edition.
2. R. Larson, R. P. Hostetler, B. H. Edwards, *Calculus with Analytic Geometry,* 7th edition.