

VISUALIZATION OF MÖBIUS TRANSFORMATIONS IN TWO AND THREE DIMENSIONS USING A CAS

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Introduction

We have created several activities for undergraduates using Derive to illustrate the beautiful geometric behavior of Möbius transformations. The purpose of this short note is to briefly describe the key aspects of these activities and how Derive is used in these activities. The activities are designed to meet several important goals: geometric visualization, hands-on activities, reinforcement of basic mathematics, and the establishment of connections between several areas of mathematics.

A wonderful aspect of Derive as a pedagogical tool is that it is a versatile computer algebra system that still requires the user to perform a fair amount of basic precalculus mathematics to create more sophisticated examples and explorations. The activities which are briefly described here can be presented to students at several levels. At a higher-level, students could be expected to create from scratch the Derive routines to construct the examples. At a lower-level, the instructor can provide the students with a copy of the collection of the routines in the file **mobmath.mth** [3] and more detailed instructions about constructing the examples. Even though these activities were designed for Derive, they are easily transferable to any computer algebra system with graphing capabilities.

Möbius Transformations

Let \mathbf{M} denote the group of Möbius transformations $z \mapsto \frac{az+b}{cz+d}$ acting on $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, where a, b, c, d are complex numbers such that $ad - bc = 1$. The group \mathbf{M} is the group of orientation-preserving conformal automorphisms of $\hat{\mathbf{C}}$ and is called the *Möbius group*. We can further identify \mathbf{M} with the group $\mathbf{PSL}(2, \mathbf{C})$, the group of 2×2 matrices with complex coefficients, determinant 1, modulo the equivalence relation $A \sim -A$. By using the 2×2 matrix representation for Möbius transformations, we can easily create short routines in Derive to calculate properties of the Möbius transformations and obtain parametric representations which we can plot.

In the upper halfspace model of hyperbolic three-space \mathbf{H}^3 , hyperbolic planes are either vertical planes or hemispheres which are orthogonal to its boundary at infinity, $\hat{\mathbf{C}}$. In both cases, the boundary at infinity of a hyperbolic plane is either a circle or line, which we consider as a circle through ∞ ; For simplicity, we refer to both circles and lines as *generalized circles*. Each element in \mathbf{M} is the product of an even number of reflections in circles in $\hat{\mathbf{C}}$. We can extend the action of each element in \mathbf{M} to an orientation-preserving isometry of the upper halfspace model of hyperbolic three-space \mathbf{H}^3 by reflecting in the corresponding hyperbolic planes bounded by the generalized circles in $\hat{\mathbf{C}}$ which generate the Möbius transformation. This extension is called the *Poincaré extension* and takes the following form in the upper halfspace model:

$$\tilde{f}(z, t) = \left(\frac{(az + b)\overline{(cz + d)} + a\bar{c}t^2}{|cz + d|^2 + |c|^2t^2}, \frac{|ad - bc|t}{|cz + d|^2 + |c|^2t^2} \right),$$

where $f(z) = \frac{az+b}{cz+d}$. (See [1] for details.) We thus can identify the group $\text{Isom}^+(\mathbf{H}^3)$ of orientation-preserving isometries of \mathbf{H}^3 with $\text{PSL}(2, \mathbf{C})$. Using the Poincaré extension, we use a simple user-defined routine `Poincare_Ext_Image` in Derive to obtain parameterizations of images of hyperbolic planes under Möbius transformations to geometrically investigate the three-dimensional geometric aspects of Möbius transformations.

Geometric Visualization

The primary goal of the activities is to obtain deep geometric intuition about the behavior of Möbius transformations. One of the activities, *Preservation of Circles Under Möbius Transformations* [4], is a strictly two-dimensional investigation of the classical result that Möbius transformations map generalized circles in $\hat{\mathbf{C}}$ to generalized circles in $\hat{\mathbf{C}}$ (for example, see Palka [6], p. 398). This activity uses an user-defined routine called `Plane_Circ_Image` to obtain parametric representations of *images* of circles under a Möbius transformation to visualize the aforementioned phenomenon. The activity proceeds to verify that the image of a generalized circle under a Möbius transformation is indeed a circle by constructing appropriate perpendicular bisectors of images of points, which, in turn, leads to the derivation of the Cartesian equation of a circle or a line using basic analytic geometry.

The other activities focus on developing the inter-relationship between the two- and three-dimensional behavior of Möbius transformations. The novel aspect of the investigations between the two- and three-dimensional geometry is that these activities introduce the mathematics in a suitable manner for undergraduates, whereas the three-dimensional geometry of Möbius transformations is not part of the standard undergraduate course in complex analysis and is not introduced, if at all, until graduate courses in topology or hyperbolic manifolds.

Hands-on Activities, Reinforcement, Connections

Our activities approach new mathematical content by investigating concrete examples that require the students to perform basic calculations using familiar mathematics: basic geometry, analytic geometry, trigonometry, and parametric representations. Even though Derive is used to perform the necessary calculations, the student must demonstrate a sound understanding of the basic mathematics to enter the appropriate expressions into Derive. This dependence on precalculus mathematics can help demystify the abstraction of the new mathematics as well as reinforce and re-integrate this basic mathematics in advanced courses outside the calculus sequence. In particular, these activities can provide nice illustrations of applications of basic mathematics to pre-service secondary education students that can help form deeper connections between the concepts that they will teach in high school.

These activities also illustrate connections between advanced mathematical topics. The activity *Construction of a Hyperbolic Surface and Kleinian Group* [2] not only uses complex variables and hyperbolic geometry but also integrates topology and algebra by using the Poincaré Polyhedron Theorem (see [1] or [5]), to construct the fundamental domain of discrete group actions on the hyperbolic plane and on hyperbolic three-space, by using group presentations, and by investigating the corresponding quotient manifolds. This activity also provides a constructive approach to visualizing the fact that closed surfaces of genus greater than one naturally admit hyperbolic structures (see [7]).

Sample Activity: Visualization of Hyperbolic Transformations

A *hyperbolic* transformation has two fixed points in $\mathbf{H}^3 \cup \hat{\mathbf{C}}$, both of which are in $\hat{\mathbf{C}}$ and takes the form:

$$\begin{bmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{bmatrix} : (z, t) \mapsto (\lambda z, |\lambda|t), \quad |\lambda| \neq 1.$$

The hyperbolic transformation is *purely hyperbolic* if λ is real and positive and is otherwise called *loxodromic*.

This activity illustrates the relationship between the two- and three-dimensional geometry of a hyperbolic Möbius transformation. The activity first investigates the two-dimensional aspects of a hyperbolic transformation. The investigation includes a study of representing a purely hyperbolic transformation as the product of reflections in two circles $\hat{\mathbf{C}}$ and a loxodromic transformation as the product of four reflections in circles in $\hat{\mathbf{C}}$, dilation and rotational components of the transformation, and identification of the attracting and repelling fixed points of the transformation. Additional geometric information is obtained from the three-dimensional viewpoint. A hyperbolic transformation has a geodesic axis in \mathbf{H}^3 joining its fixed points in $\hat{\mathbf{C}}$. This

axis is either a vertical line (if one of the fixed points is at ∞ in the upper halfspace model of \mathbf{H}^3) or a semicircle orthogonal to $\hat{\mathbf{C}}$ (if both fixed points are finite.) The rotational component observed in the two-dimensional investigation carries over to a rotation about the geodesic axis. The two-dimensional dilation corresponds to a three-dimensional translation along the geodesic axis.

A nice extension of the three-dimensional investigation is to construct the boundary of a regular hyperbolic neighborhood of the geodesic axis. The boundary will be a Euclidean cone with vertex at the finite fixed point and will be a banana-shaped surface converging to points at both of the fixed points. Finding the parameterization of the cone-shaped surface requires some analytic geometry and trigonometry, whereas the construction of the banana-shaped surface requires more trigonometry and some understanding of conjugation in the Möbius group. A hyperbolic transformation γ generates a discrete, infinite cyclic subgroup Γ of the Möbius group. A further extension of the activity investigates the relationship between the a fundamental domain F of the action of Γ on \mathbf{H}^3 and a fundamental domain $D = \partial F$ of the action of Γ on $\hat{\mathbf{C}} \setminus \{\text{fixed points of } \gamma\}$. From this we can observe that $[\mathbf{H}^3 \cup (\hat{\mathbf{C}} \setminus \{\text{fixed points of } \gamma\})]/\Gamma$ is manifold with boundary $(\hat{\mathbf{C}} \setminus \{\text{fixed points of } \gamma\})/\Gamma$.

Derive Routines for Möbius transformations

We have developed a package of Derive routines called **mobmath.mth** [3], which can be obtained from the author. Below we provide two routines, which allow the Derive user to obtain parameterizations of images of circles, and (hemispherical) hyperbolic planes. The routine `Plane_Circ_Image(r, a, b, M)` returns the image of the parametric representation $(r \cos t + a, \sin t + b)$ of the circle of radius r centered at the point (a, b) in the plane under a 2×2 matrix representation of a Möbius transformation M .

```
Plane_Circ_Image(r, a, b, M) :=
  Prog
    z :=
      f1(z) := (ELEMENT(M, 1, 1)·z + ELEMENT(M, 1, 2))/(ELEMENT(M, 2, 1)·z
      + ELEMENT(M, 2, 2))
      z := r·COS(t) + a + i·r·SIN(t) + b)
      [Re(f1(z)), IM(f1(z))]
  RETURN v2
```

The second routine called `Poincare_Ext_Image(M, vec)` produces the image of the ordered triple `vec` under a 2×2 matrix representation of a Möbius transformation M . To study basic three-dimensional geometric properties of a Möbius transformation, we often take `vec` to be a parameterization for a hemisphere whose boundary circle lies in the complex plane, which is a hyperbolic plane in the upper halfspace model for \mathbf{H}^3 .

```

Poincare_Ext_Image(M, vec) :=
  Prog
     $\alpha$  := ELEMENT(vec, 1)
     $\beta$  := ELEMENT(vec, 2)
     $\gamma$  := ELEMENT(vec, 3)
     $z := x + i \cdot y$ 
     $g(x, y, t) := \text{RE}(((A \cdot z + B) \cdot \text{CONJ}(C \cdot z + D) + A \cdot \text{CONJ}(C) \cdot t^2) / (\text{ABS}(C \cdot z + D)^2 + \text{ABS}(C)^2 \cdot t^2))$ 
     $h(x, y, t) := \text{IM}(((A \cdot z + B) \cdot \text{CONJ}(C \cdot z + D) + A \cdot \text{CONJ}(C) \cdot t^2) / (\text{ABS}(C \cdot z + D)^2 + \text{ABS}(C)^2 \cdot t^2))$ 
     $l(x, y, t) := \text{ABS}(A \cdot D - B \cdot C) \cdot t / (\text{ABS}(C \cdot z + D)^2 + \text{ABS}(C)^2 \cdot t^2)$ 
     $p := [g(\alpha, \beta, \gamma), h(\alpha, \beta, \gamma), l(\alpha, \beta, \gamma)]$ 
    RETURN p

```

For Further Information

Complete activities and the latest version of **mobmath.mth** can be obtained by contacting the author directly at tcomar@ben.edu.

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References

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