Fractals for the Classroom on Calculators and Computers

by

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Abstract

Fractals are contemporary geometric notions within Chaos theory that are accessible to students at almost any level and can be used to generate interest and enthusiasm among students of all ages. One of the most familiar fractals is the Sierpinski Triangle. I will describe how this triangle can be generated, and I will demonstrate this on the TI-83 Plus SE. The resulting figure consists of an infinitely repeating, self-similar pattern of three smaller triangles forming the larger one. The ‘compression ratio’ (the size of the smaller triangles as compared to the larger one) for this Sierpinski Triangle is one half.

Similar processes can be used with six points and a compression ratio of one third to generate a Sierpinski Hexagon or five points and a compression ratio of three eights to generate a Sierpinski Pentagon. Other interesting images can be formed by using various rotations of the points about the vertices. Examples of these types of fractals will also be demonstrated with the calculator.

Finally I will reference a website designed by Robert Devaney as part of the Dynamical Systems and Technology project at Boston University. This site can produce fractals in color with a great deal of speed and flexibility.

Fractals for the Classroom on Calculators and Computers

Introduction

Most of the mathematics that we teach our students throughout high school and even in college is hundreds, if not thousands of years old. Is it any wonder that many of our students consider mathematics to be stale and boring? Why are we surprised when they are not interested in a subject that we find so fascinating and when they don't see its relevance to their modern day lives?

In particular, I would like to consider the field of geometry. All of the geometry that we teach in high school is contained in Euclid's 'Elements', which was written over 2000 years ago. I teach a course in geometry to our elementary education majors that consists mainly of geometric content appropriate for grades K - 4. We teach this course in the manner recommended by the 'Standards' (NCTM, 2000) with many hands-on activities and projects designed to encourage the students to become active learners. In my efforts to make this course interesting and relevant to their lives, I have incorporated some of mathematical ideas developed in the field of Chaos.

Fractals

Chaos theory presents us with some exciting, contemporary mathematics that can be used to capture the attention of and even develop enthusiasm in students of all ages. Within Chaos theory, fractals are modern geometric notions that are accessible to students at almost any level. Students can create their own fractals with various types of classroom activities, and they can generate them on their computers and calculators. Many students become so fascinated with these patterns that they develop projects and enter science fair competitions based on these notions.

One of the most familiar fractals is the Sierpinski triangle. This triangle can be generated as follows. Begin with three points in the Cartesian Plane and choose any point within the triangle formed by these vertices as a starting point or seed. Randomly select any of the three original vertices and move halfway toward that vertex. Mark this point and then repeat this process using this last point as the new seed. Of course this is a genuinely random process and the sequence of points generated is different each time; yet the resulting figure is always the same, Sierpinski's triangle.

Sierpinski's triangle can be use to illustrate many geometric ideas, such as linear measure, area, perimeter, etc. In general, the figure consists of an infinitely repeating, self-similar pattern of three smaller triangles that form the larger one. Each of the smaller triangles is one half the size of the original triangle in terms of linear dimensions. This is a result of moving one half of the distance to the vertices each time, a fact that we will refer to as a 'compression ratio' of one half. I usually demonstrate Sierpinski's triangle to the students in my Geometry for Elementary Education Majors class by using a program on a TI-83 calculator with an overhead view screen (see Appendixes A and B). The students are generally very impressed by this fractal image. They are amazed by this demonstration of predictability among randomness which is at the heart of Chaos theory.

There are other examples of fractal images that can be generated in a similar manner. For example, the triangle program can be modified to produce what could be called Sierpinski's pentagon or Sierpinski's hexagon. The hexagon results from choosing six points as the original vertices, picking an arbitrary starting point (or seed), and moving two thirds of the distance toward a randomly chosen vertex. This gives a compression ratio of one third, that is, the distance from the moving point to the vertex is compressed by a factor of three. The resulting fractal consists of six self-similar hexagons each of which is one third the size of the original. The Sierpinski pentagon results from an initial five points and iterations performed with a compression ratio of three eights. This process produces five self-similar pentagons each of which is three eights the size of the original (Devaney, 1995).

I demonstrate these fractals with TI-83 programs (see Appendix A and B), but these are necessarily more complicated. Thus they take longer to produce the fractals and the resolution available on the TI-83 does not produce images of as fine a quality as it does for the triangle. I might mention here that these minor drawbacks do not seem to bother the students in any way.

There are many more interesting geometric patterns that can be created by introducing rotations into the previously described processes. For example suppose that we consider the basic Sierpinski triangle but introduce a rotation into the process. Let's select the top vertex and rotate any point that is moved toward that vertex 180° about that vertex. Any point that is moved to one of the bottom vertices will be done in the usual manner. This produces a fractal triangle that is similar to but different from the original Sierpinski's triangle. In this case the top self-similar copy is turned upside down. I also demonstrate this triangle on the calculator (see Appendixes A and B).

Of course the selection of the top vertex in the previous example was complete arbitrary as was the choice of 180° for the amount of the rotation. In fact we could pick any vertex, or combination of vertices, and choose any size angle(s) for the rotation. The rotations could even be different sizes and directions for the different vertices. In this manner it is possible to create many different fractal images from just a few basic geometric shapes (e.g., I have used only the triangle, pentagon, and hexagon here).

Conclusion

There are many advantages to using a graphing calculator such as the TI-83 to explore these images in a mathematics classroom. In general research (O'Callaghan, 2002) shows that the use of technology can enhance students' understanding of mathematical concepts. In this case there are many ways to use this process and these images to help students to improve there spacial visualization skills as well as their overall understanding of geometry and mathematics. The calculator is mobile and relatively inexpensive; so that all students can have one and can use their imagination and creativity to explore various possibilities with this patterns.

There are also some drawbacks however. The calculators are relatively slow in creating the fractals (although I like this feature because it is very interesting to watch the students try to determine the pattern as it gradually takes shape on the screen). The new TI-83 Plus SE is about three times as fast as the original TI-83 and has improved this situation considerably. The resolution on the calculator screen is limited, and this detracts somewhat from the clarity of the images. Finally the TI-83 does not allow for the use of color that can make the fractals much more attractive and interesting.

All of these drawbacks are minor and present no obstacles to the use of these fractals to aid the teaching and learning of mathematics. However there is a website that eliminates all of these unfavorable characteristics. The site (http://math.bu.edu//DYSYS/dysys.html) was designed by Robert Devaney as part of the Dynamical Systems and Technology project at Boston

University. This site has many fascinating features, but the activity that refers most closely to the ideas presented in this paper is called ‘Fractalina’ (see Appendix C). This applet allows the user to choose the number of vertices, locate them anywhere in the plane, and define the compression ratios and rotations for each vertex. It then displays the resulting fractal almost instantly in the colors chosen by the user.

References

Devaney, R. L. (1995, June). Explorations in the Chaos Club. *FOCUS: The Newsletter of the Mathematical Society of America, 15(3),* 8-9.

O’Callaghan, B.R. (2002). Students’ conceptual Knowledge of Functions. In J. Sowder & B. Schappelle (Eds.), *Lessons Learned from Research (pp 207-208).* Reston, Va.: NCTM.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, Va.: NCTM.

Texas Instruments. (1993). *TI-82 Graphics Calculator Guidebook (p 14-8)*. Dallas, TX.

Appendix A

```
Program Sierpins (TI, 1993)
(This program generates
Sierpinski's triangle)
FnOff :ClrDraw
PlotsOff
AxesOff
0→Xmin:1→Xmax
0→Ymin:1→Ymax
rand→X:rand→Y
For(K,1,3000)
rand→N
If N≤1/3
Then
.5X→X
.5Y→Y
End
If 1/3<N and N≤2/3
Then
.5(.5+X)→X
.5(1+Y)→Y
End
If 2/3<N
Then
.5(1+X)→X
.5Y→Y
End
Pt-On(X,Y)
End
StorePic 3
```

```
Program Sierpin1
(This program generates
Sierpinski's triangle with 180°
rotation about the top vertex)
FnOff :ClrDraw
PlotsOff
AxesOff
0→Xmin:1→Xmax
0→Ymin:1.5→Ymax
rand→X:rand→Y
For(K,1,3000)
rand→N
If N≤1/3
Then
.5X→X
.5Y→Y
End
If 1/3<N and N≤2/3
Then
.5(.5+X)→X
.5(1+Y)→Y
-Y+2→Y
-X+1→X
End
If 2/3<N
Then
.5(1+X)→X
.5Y→Y
End
Pt-On(X,Y)
End
StorePic 4
```

```
Program Sierpen
(This program generates
Sierpinski's pentagon)
FnOff :ClrDraw
PlotsOff :AxesOff
0 →Xmin:3.5 →Xmax
0 →Ymin:3.5 →Ymax
rand →X:rand →Y
For(K,1,3000)
rand →N
If N≤1/5
Then
5/8(.6-X)+X →X:3/8Y →Y
End
If 1/5<N and N≤2/5
Then
5/8(2.6-X)+X →X:3/8Y →Y
End
If 2/5<N and N≤3/5
Then
5/8(3.2-X)+X →X:5/8(1.9-Y)+Y →Y
End
If 3/5<N and N≤4/5
Then
5/8(1.6-X)+X →X:5/8(3.2-Y)+Y →Y
End
If N>4/5
Then
3/8X →X:5/8(1.9-Y)+Y →Y
End
Pt-On(X,Y)
End
StorePic 5
```

```
Program Sierphex
(This program generates
Sierpinski's hexagon)
FnOff :ClrDraw
PlotsOff :AxesOff
0 →Xmin:4 →Xmax
0 →Ymin:4 →Ymax
rand →X:rand →Y
For(K,1,3000)
rand →N
If N≤1/6
Then
2/3(1-X)+X →X:1/3Y →Y
End
If 1/6<N and N≤1/3
Then
2/3(3-X)+X →X:1/3Y →Y
End
If 1/3<N and N≤1/2
Then
2/3(4-X)+X →X:2/3(2-Y)+Y →Y
End
If 1/2<N and N≤2/3
Then
2/3(3-X)+X →X:2/3(4-Y)+Y →Y
End
If 2/3<N and N≤5/6
Then
2/3(1-X)+X →X:2/3(4-Y)+Y →Y
End
If N>5/6
Then
1/3X →X:2/3(2-Y)+Y →Y
End
Pt-On(X,Y)
End
StorePic 6
```

Appendix B

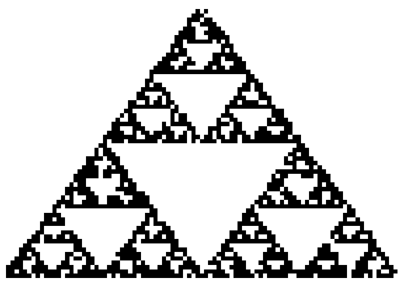


Figure 1. Sierpinski's Triangle.

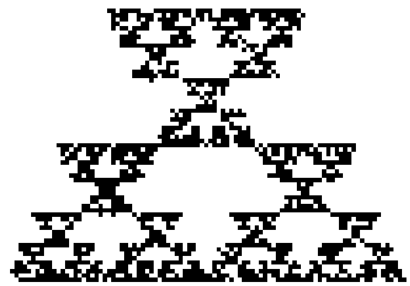


Figure 2. Sierpinski's Triangle with a 180° Rotation about the top vertex.

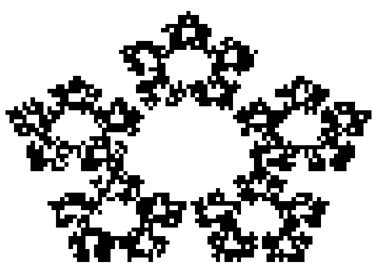


Figure 3. Sierpinski's Pentagon.

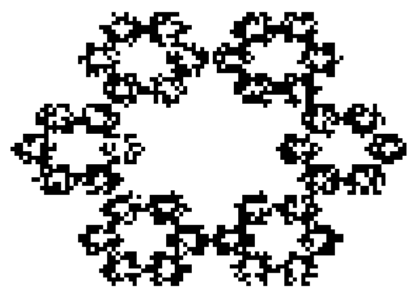
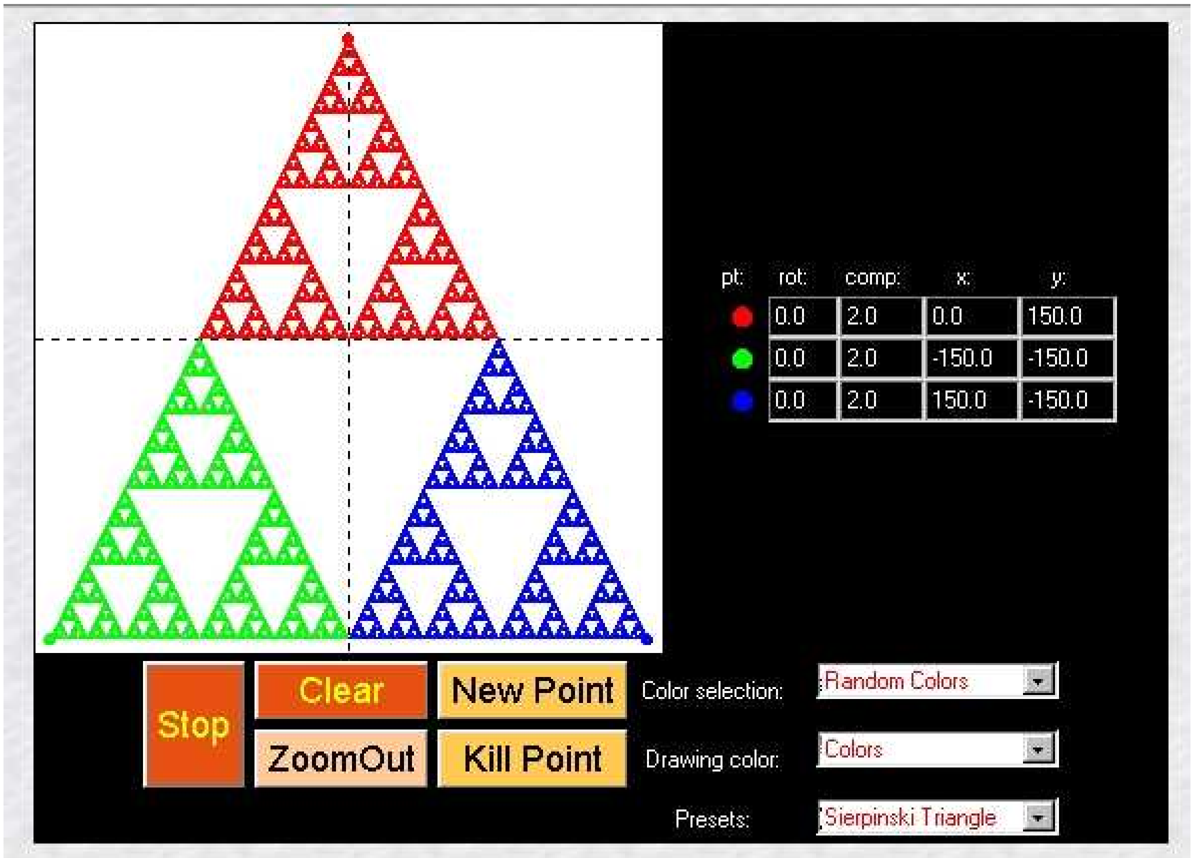


Figure 4. Sierpinski's Hexagon.

Appendix C

Fractalina



Biographical Sketch

Brian R. O'Callaghan is an associate professor of mathematics at Southeastern Louisiana University. His primary interests include the use of technology in teaching mathematics and students' understanding of mathematical concepts. He works with pre-service teachers at the elementary, middle and secondary levels. He received his Ph.D. in mathematics education at Louisiana State University in 1994.