Quaternions and Rotations in 3-Space, With the TI-83

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The Complex Numbers, and the Quaternions are the only possible associative division algebras over the reals. The Complex Numbers can be viewed as a 2-dimensional vector space over the reals, while the Quaternions are a 4-dimensional real vector space. Multiplication by a unimodular complex number is, essentially, rotation (in the plane) through the angle (amplitude) of the unimodular complex multiplier. Analogously, there is a multiplication operation, though with a slight twist, by unimodular quaternions which accomplishes a rotation in 3-space.

If **v** is a vector in 3-space, then **v** can be viewed as a pure quaternion, and if **q** is a unimodular quaternion, then rotation of **v** through an angle determined by **q**, and about an axis, also determined by **q**, is given by

$$\mathrm{R}_q\colon\ \mathbf{v} \longrightarrow \mathbf{q}\mathbf{v}\mathbf{q}^{-1}.$$

The purpose of this short note is to develop a calculator program (for the TI-83) that will perform the transformation above. And this may be worthy of some slight consideration because the TI-83 does not have symbolic algebra capabilities. Rotations in 3-space, using quaternions, is favored, in applications, over the traditional linear algebra approach, using 3 by 3 matrices, because of the savings in computer storage. It is much cheaper to store the components of the quaternion **q** than it is to store the 9 entries of a 3 by 3 matrix! Besides, the quaternion's components display the rotation axis, as well as the angle of rotation.

After having said that, there is a bit of irony in the approach we take here, since we will rely on a matrix representation of Q (the quaternions) in order to effect the transformations (rotations). But, of course, our objective is not savings. Rather, we only seek to be able to exploit the hand-held calculator as far as is possible.

Quickly reviewing the multiplication in Q : We use the standard basis, $1, i, j, k,$ where

$$i^2 = j^2 = k^2 = -1, \text{ and}$$

$$ij = k, \;\; jk = i, \; ki = j, \text{ and}$$

$$ji = -k, \;\; kj = -i, \;\; ik = -j \quad \text{(anti-commutativity).}$$

Extending to all of Q, by linearity and associativity, we get a 4-dimensional algebra over the reals. It is customary to write quaternions in the form

$$(s,x,y,z)$$

to stand for

$$s*1+x*i+y*j+z*k.$$

We use the $*$ here to emphasize the operation of multiplying a quaternion by a real number. This shall be suppressed in the rest of this note. It is also customary to express the quaternion (s,x,y,z) as

$$< s, \;\; \mathbf{v} >,$$

with s referred to as the scalar part, and $\mathbf{v}$ the vector part, with vector components x,y,z. If $s = 0$, we say that we have a pure quaternion. The pure quaternions are sometimes identified with R^3. It is now easy to verify that

$$(s,x_1,y_1,z_1)(w,x_2,y_2,z_2) =< s,\mathbf{v}_1 > \;\; < w, \; \mathbf{v}_2 > \;\; = \;\; < sw - \mathbf{v}_1 \bullet \mathbf{v}_2, \; s\mathbf{v}_2 + w\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2 >$$

where the operations $\bullet$ and $\times$ are the familiar dot product, resp., cross product, in R^3.

When $q = (s,x,y,z)$, we define $\|q\| = \sqrt{s^2+x^2+y^2+z^2}$, the modulus of q. And, of course, if $\|q\| = 1$, we say that q is unimodular. In this case, it is possible to express q as

$$q = \;\; < s,\mathbf{v} > \;\; = \;\; < \cos(\frac{\theta}{2}), \; \sin(\frac{\theta}{2})\hat{\mathbf{v}} >,$$

a sort of polar form for q, where $\hat{\mathbf{v}}$ is a unit vector in the direction of $\mathbf{v}$, the axis of the rotation, and θ is the angle through which we rotate.

Now take $q = \; < s, \; \mathbf{v} > \; = \; < \cos(\frac{\theta}{2}), \; \sin(\frac{\theta}{2})\hat{\mathbf{v}} >$, a unimodular quaternion, as above, and suppose that $\mathbf{w}$ is any vector in R^3. It is fairly straightforward to verify that the transformation

$$\mathrm{R}_q : \mathbf{w} \longrightarrow q\mathbf{w}q^{-1}$$

is a linear transformation in R^3; that $\mathbf{v}$ is an eigenvector belonging to the eigenvalue 1; that the plane through the origin, orthogonal to $\mathbf{v}$, is an invariant subspace, and that the transformation, restricted to this plane, is a rotation through the angle θ. This is all well known, and an excellent treatment of these ideas may be found in Chapter Six of [1].

For this note, we shall be content to develop a calculator (TI-83) program to perform the rotation transformation. We consider the right regular representation of Q, in which

$$1 ---> \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E$$

$$i ---> \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = A$$

$$j ---> \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = B$$

$$k ---> \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = C$$

Then the quaternion

$$q = (s,x,y,z) ---> sE + xA + yB + zC =$$

$$\begin{bmatrix} s & x & y & z \\ -x & s & -z & y \\ -y & z & s & -x \\ -z & -y & x & s \end{bmatrix}.$$

With the TI-83, we can construct, and store, the matrices E, A, B, and C above. The following program should be self-explanatory:

The Program

PROGRAM: ROTATION
:Input "SCALAR:", S
:Input "[A]-COEFF:", A
:Input "[B]-COEFF:", B
:Input "[C]-COEFF:", C
:$\sqrt{(S^2 + A^2 + B^2 + C^2)}$ —>W
:(S/W) $-- >$ S
:(A/W) $-- >$ A
:(B/W) $-- >$ B
:(C/W) $-- >$ C
:Input "1ST:", X
:Input "2ND:", Y
:Input "3RD:", Z
:$S * [E] + A * [A] + B * [B] + C * [C] -- > [G]$
: $X * [A] + Y * [B] + Z * [C] -- > [F]$
:$[G] * [F] * [G]^{-1} - - > [H]$
:Disp $[H]$
:Pause
:2*arccos(S) $-- >$ L
:Disp "ANGLE IS", L,"*RADIANS*"
:Disp "ANGLE IS", $L * 180/\pi$,"DEGREES"

Note that the first row of each matrix representing a quaternion contains the components of the quaternion in the proper order, and the first column is the conjugate of that quaternion.

References

1. Needham, Tristan, *Visual Complex Analysis*,