AN INTERESTING PROBLEM FROM CALCULUS – AND MUCH MORE

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1. INTRODUCTION. In the latest Calculus; by J. Stewart, we can notice many new problems which can be solved by using modern mathematical technologies, for example, Mathematica and Geometer's Sketchpad. Solving these problems with the help of these programs is not only more enjoyable, but in some cases may lead to profound generalizations and discoveries.

In this paper we will concentrate on one of such problems, the problem #78, on page 70. This problem is stated there as follows: **The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point (0, r), Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0+$?**

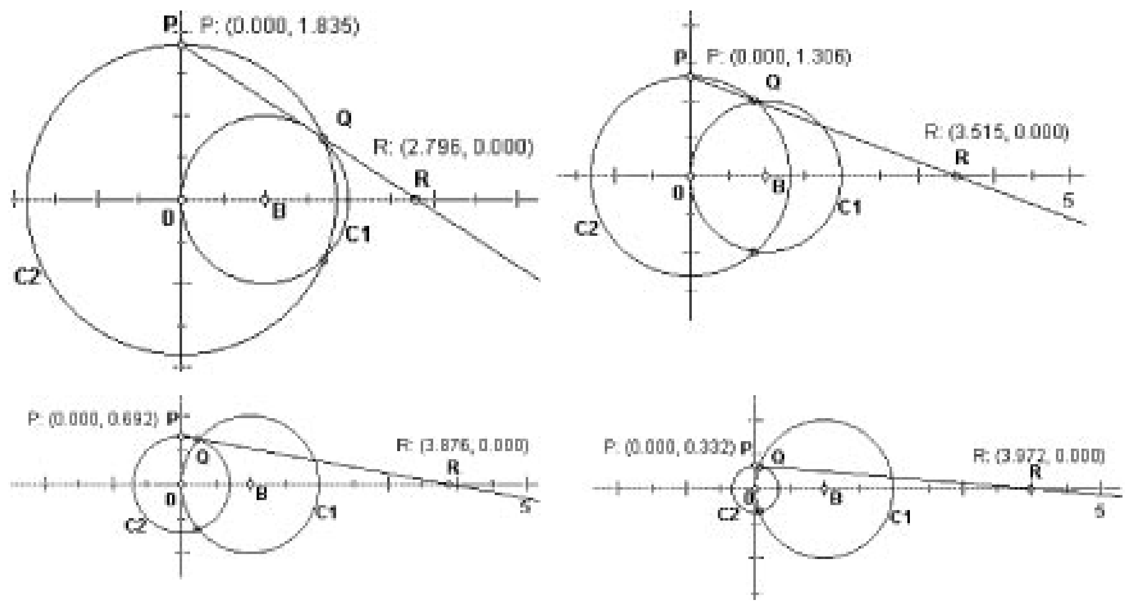

Figure 1. The illustration of the above problem.

To get an idea about the solution, we decided to use the Geometer's Sketchpad, Version 3.0. The **Figure 1** shows four different snapshots taken form Sketchpad in succession of r decreasing to 0. To create these pictures in Sketchpad is a standard procedure. With the

Sketchpad, we can keep shrinking continuously the circle C_2 by dragging with mouse the point P towards origin, and at the same time observing that the point R approaches the number 4. Furthermore, by taking advantage that the drawings in the Sketchpad are dynamic, we can easily change the circle C_1 to a circle of any radius r, and make an observation that the point R approaches the number $4 \cdot r$.

A similar exercise can be carry out for the problem in which the circle C_1 is replaced be a straight line with positive slope passing through origin. This time we will observe that the point R approaches 0.

2. GENERALIZATION. The described above problem can be easily generalized by replacing the fixed circle C_1 by any curve. In the **Figure 2** below, this curve is denoted again by $\mathbf{C_1}$. Since we are going to shrink the circle C_2, the only important part of the curve is the part that is immediately close to origin. Therefore, naturally we can assume that the curve C_1 passes through the origin, is smooth enough; and for positive x close to origin, it lies in the first quadrant, and is increasing there.

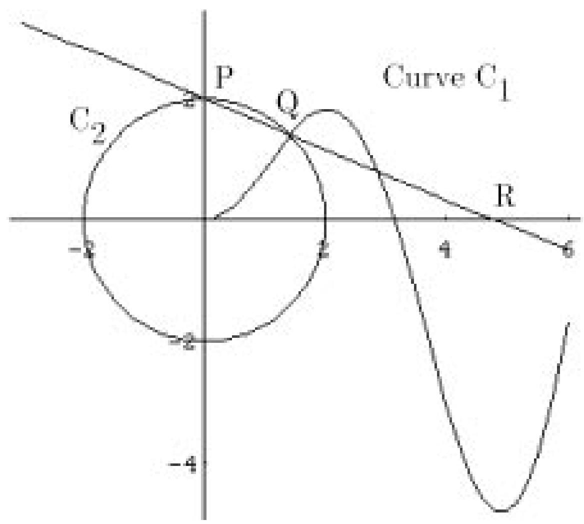

Figure 2.

Thus our problem, with these assumptions, is to **find limit of the point R as radius r of circle C_2 approaches 0**.

3. SETTING UP FOR GENERAL SOLUTION. In this paragraph, We will refer to the notation of the Figure 2. If the curve C_1 is given by an equation $F(x, y) = 0$, then the coordinates of the point $Q = (x, y)$, the point of intersection of the circle C_2 and the curve C_1, can be found by solving system of equations,

$$F(x, y) = 0 \text{ and } x^2 + y^2 = r^2 \tag{3.1}$$

Then, the equation of the line through the points P and Q is

$$Y - r = \frac{Y - r}{x} X . \tag{3.2}$$

Setting $Y = 0$ in the equation (3.2), we get the x-coordinate of the point R,

$$X = \frac{rx}{y - r} . \tag{3.3}$$

Thus, our problem is to find

$$\lim_{r \to 0+} \frac{rx}{y - r} , \tag{3.4}$$

which we denote by limtR.

However, this problem as simple as it might seem at the first sight, to carry out this computation, even for simple equations $F(x, y) = 0$, is very laborious and tedious, or even impossible to do especially for most students. There are three main reasons for these, the system of equations (3.1) is to difficult or impossible to solve, the expression (3.3) is lengthy, and thus the computation of limit (3.4) is not clear. In contrast, with the Mathematica all these difficulties might be avoided, especially if the equation $F(x,y) = 0$ is relatively simple. In this paper we decided restrict ourselves to the cases when the equation $F(x,y) = 0$ represents conics. We also must remember that our goal is not to do the computations, but make a discovery.

4. MATHEMATICA IN ACTION. We will apply the Mathematica to do all the computations described in the paragraph **3**. To do these we can use the following program.

pointQ:=Solve[{x^2 + y^2 = =r^2, F(x,y) = =0}, {(x, y}]

pointR:=(r*x)/(r − y)/. {pointQ[[4]][[1]], pointQ[[4]][2]]}

limitR= Limit[pointR, r − >0}

Warning: For some reason, if the equation $F(x,y) = 0$ have a parameter, the Limit command produces incorrect output. For example, for parabola given by equation $y^2 = 2$ a x, the output is 0, which is incorrect. However, if we define the parameter a to be the number E or Pi, the Limit is computed correctly, and at the same time we can see the answer in general form since E and Pi being transcendental numbers will not cancel out during symbolic computation. Thus, in the above program we can line a: = Pi, and so on.

The intermediate results of this computation we will illustrate for parabola $y^2 = 2ax$ only, the final results for the other curves are included in **Table 1**, the middle column.

In[1]:=

PointQ=Solve[{x^2 + y^2==r^2, y^2==2 a x},{x,y}]

Out[1]=

$\{\{x -> \frac{-2\ a^2 - 2\ Sqrt[a^4 + a^2\ r^2]}{2\ a}$, $y -> -\ Sqrt[-2\ a^2 - 2\ Sqrt[a^4 + a^2\ r^2]]\},$

$\{x -> \frac{-2\ a^2 - 2\ Sqrt[a^4 + a^2\ r^2]}{2\ a}$, $y -> Sqrt[-2\ a^2 - 2\ Sqrt[a^4 + a^2\ r^2]]\},$

$\{x -> \frac{-2\ a^2 + 2\ Sqrt[a^4 + a^2\ r^2]}{2\ a}$, $y -> -\ Sqrt[-2\ a^2 + 2\ Sqrt[a^4 + a^2\ r^2]]\},$

$\{x -> \frac{-2\ a^2 + 2\ Sqrt[a^4 + a^2\ r^2]}{2\ a}$, $y -> Sqrt[-2\ a^2 + 2\ Sqrt[a^4 + a^2\ r^2]]\}\}$

In[2]:=

pointR=(r x)/(r − y)/. { pointQ[[4]][[1]], pointQ[[4]][[2]]}

Out[3]=

$$\frac{r(-2\ a^2 + 2\ Sqrt[a^4 + a^2\ r^2])}{2a\ (r - Sqrt[-2\ a^2 + 2\ Sqrt[a^4 + a^2\ r^2]])}$$

In[4]:=

a:=Pi

In[5]:=

limitR=Limit[pointR, r − >0]

Out[5]=

4 Pi

5. LITTLE DISCUSSION. Before we show the results of our computation, we would like to discuss the possible solutions. Our first guess is that limit of R depends on the tangent line to the curve C_1 at origin. However, this assertion has to be rejected since for any line passing through the origin the point R tends to the origin. The second guess is that this limit should depend on the curvature of curve C_1 at origin, since the curvature fully characterizes a curve. Keeping these remarks in our mind, the analysis of the **Table 1** should bring the desired discovery

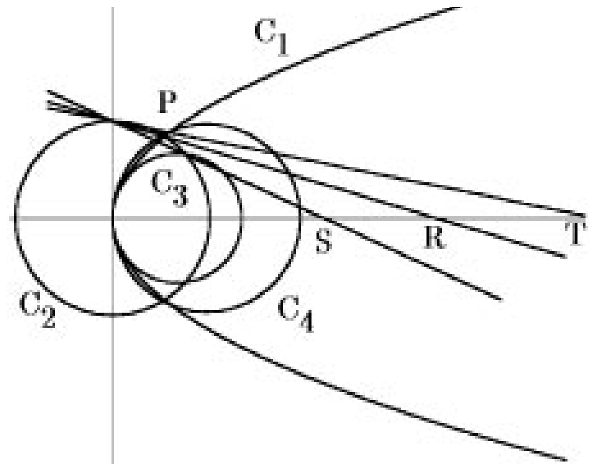

Figure 3. Illustration of the main Theorem.

Curve	Limit of R	Curvature at Origin
Line $y = ax$	0	0
Circle $(x - r)^2 + y^2 = r^2$	4r	$\frac{1}{r}$
Ellipse $\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{4b^2}{a}$	$\frac{a}{b^2}$
Hyperbola $\frac{(x+a)^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{4b^2}{a}$	$\frac{a}{b^2}$
Circle $(x - a)^2 + (y - b)^2 = a^2 + b^2$	0	$\frac{1}{\sqrt{a^2+b^2}}$

Table 1.

6. MAIN DISCOVERY. Under the assumptions of Paragraph 2, observations made out of the **Table 1**, and notations of **Figure 3**, we can state the following

THEOREM. If the curvature circle C_3 of a curve C_1 at origin has radius ρ and its center lies on x-axis, then the point R approaches number 4 ρ. Otherwise, the point R approaches origin.

Proof. We will give only the proof for the case when the center of curvature lies on x-axis, this is illustrated by **Figure 3**. For the other case the proof is the same.

Let $\epsilon > 0$ be any real number, and C_4 be the circle with radius $\rho + \epsilon$ and center $(\rho + \epsilon, 0)$. The points S and T are the x-intercepts of the lines passing through the point P and the intersection points of the circle C_2 with the circles C_3 and C_4, respectively. Then, since we assumed that the curve C_1 is concave down in the first quadrant, close to origin, we observe that the point R is between the points S and T. From the **Table 1**, we know that S and T tend to 4ρ and $4(\rho + \epsilon)$ respectively, as r – >0 . Therefore, limit of R is also between 4ρ and $4(\rho + \epsilon)$. Since ϵ is any real positive number, limit of R must be 4ρ.

REFERENCES

1. James Stewart, *Calculus; Early Transcendentals*, 3rd edition, Brooks/Cole, 1994.