USING SPREADSHEETS AND DERIVE TO TEACH DIFFERENTIAL EQUATIONS

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Spreadsheets, although initially designed for business applications have found recent popularity in mathematics. In the past it has been difficult, if not impossible, to do much in a one semester Differential Equations course with classical methods of approximating solutions. Calculating fourth order Runga-Kutta approximations with a calculator is so tedious that it is rarely instructive. Students feel no sense of accomplishment when they get a numerical result they find difficult to interpret. However, with a spreadsheet program students can easily, without advanced computer experience, learn how to set up any explicit finite difference approximation, graph the result, experiment with different step sizes and compare to an analytic solution.

Another sticking point in the first semester Differential Equations course are solutions expressed in terms of definite integrals which must be approximated. Students do not believe that, for example, when asked to solve the initial value problem:

$$y'' - y = 1/x; \; y(1) = 0 \; y'(1) = -2 \quad [1, \text{pg. } 188]$$

that

$$y(x) = e^{1-x} - e^{x-1} + \frac{e^x}{2} \int_1^x \frac{e^{-t}}{t}\, dt - \frac{e^{-x}}{2} \int_1^x \frac{e^t}{t}\, dt$$

is an acceptable answer and every bit as legitimate a function of x as sin(x). However, even a relatively simple to learn computer algebra package like DERIVE can graph functions defined this way. In this instance, a picture is truly worth a thousand words. Students who see a graph of this function are much more inclined to believe than ones who are shown how to use Simpson's rule to compute a few values. In fact, using DERIVE they can do both.

In this paper directions for programming a spreadsheet to approximate solutions to first order differential equations will be given and their efficacy in teaching will be discussed. Also use of DERIVE to graph direction fields, to approximate solutions, to graph functions defined by definite integrals and to help with tedious algebra will be discussed.

Programming Finite Difference Approximations in LOTUS

The easiest way to learn how to approximate solutions to differential equations on a spreadsheet is to run through an example. The following instructions are given for approximating the

solution to the differential equation (*) $y' = x^2 - y$; $y(-5) = -5$. They can be modified as explained in *6 below to approximate solutions to almost any Differential Equation of the form $y' = f(x,y)$ $y(x_0) = y_0$, provided the approximation method will work. With similar instructions and minimal instructor assistance, students in a differential equations class can learn to program spreadsheets to perform any finite difference approximation. They can graph the approximate solutions and investigate the effect of changes in step size or parameter values quickly and easily. With this tool it is possible to incorporate approximation techniques into a one semester first Differential Equations course in a meaningful way. Figure 1 below shows the formulae to be entered in LOTUS to use Euler's Method to approximate (*). Directions for entering them follow.

	A	B	C	D	E	F	G
1	**n**	**xn**	**yn**	**f(xn,yn)**	**h**	**Yn True**	**Error**
2	**1**	**-5**	**-5**	**+B2^2-C2**	**0.5**	**+B2*(B2-2)+2+$E$4*@EXP(-B2)**	**+F2-C2**
3	**+A2+1**	**+$B$2+A2*$E$2**	**+C2+$E$2*D2**	**+B3^2-C3**	**C**	+B3*(B3-2)+2+$E$4*@EXP(-B3)	+F3-C3
4	+A3+1	+$B$2+A3*$E$2	+C3+$E$2*D3	+B4^2-C4	**(C2-B2*(B2-2)-2)*@EXP(B2)**	+B4*(B4-2)+2+$E$4*@EXP(-B4)	+F4-C4
5	+A4+1	+$B$2+A4*$E$2	+C4+$E$2*D4	+B5^2-C5		+B5*(B5-2)+2+$E$4*@EXP(-B5)	+F5-C5
6	+A5+1	+$B$2+A5*$E$2	+C5+$E$2*D5	+B6^2-C6		+B6*(B6-2)+2+$E$4*@EXP(-B6)	+F6-C6

Figure 1: Lotus Formulae for Euler's Method on $y' = x^2 - y$; $y(-5) = -5$

*1   Start LOTUS and Choose 1-2-3. A Matrix with headings A B C . . . on the top and 1 2 3 ... on the left should appear. Each location within the matrix is a CELL referenced by the column letter and row number. For example the number 0.5 is in cell E2 above. You can move around the matrix using the arrow keys. The current cell is highlighted on your screen.

*2   Type the **bold** numbers, headings and formulae as they appear above (to enter $B$2 type B2 and press the F4 key, similarly for $E$4.) into cells A1, B1,...G1; A2, B2, ...,G2 and A3, B3, C3, E3 and E4. When you enter formulae what should appear in the cell is the result of the computation, not the formula. When the cell is highlighted the formula should appear above the command line which is above the matrix. (If the formula appears in the cell check to see if there is a quote at the beginning of the formula as it appears on the command line. Delete the quote and make sure the formula begins with an operand or a number.)

*3   Use the COPY command as follows: move to cell A3. Type **/** then **c**. A command line should appear above the matrix asking FROM where you wish to copy and showing (as a default) the current cell location. Hit <**enter**>. The command line then prompts you for the cell(s) TO which you wish to copy. Type **A4..A102** and hit <**enter**>. The formulas should be copied into cells A4 through A102. Now move to cell B2. Type **/** then **c**. Hit <**enter**>. The command line then prompts you for the cell(s) TO which you wish to copy. Move the cursor

to cell B3 and hit the period key. Then move the cursor down the B column. The cells should be highlighted as you move and their locations should appear on the command line. Cursor to cell B102 then hit enter. The formulas should be copied into cells B3 through B102. Repeat either copying process with C3 into C4..C102, D2 into D3..d102, F2 into F3..F102 and G2 into G3..G102. The formulae above that are not in **bold** should now be in the appropriate cells as well.

Now column C should contain the result of Euler's method on (*) with a step size of h=.5, column F should contain the true solution for the same x values and column G should contain the error in the approximation at each of these x values.

*4   To graph the approximate solution on [-5,5], type **/ G** for graph, **T** for type, and **X** for XY. Then choose to set the X data range by typing **X** at the graph menu. Highlight the values -5 through 5 in column B, using the period to mark the start of your range as you did when copying into column B in *3, then hit <**enter**>. Similarly set the A data range by typing **A** and highlighting the entries in the yn or C column next to those from -5 to 5 in the xn or B column. Type **V** to view the graph. Repeatedly pressing the escape key, until the command line is blank, will return you to the matrix. You can graph the approximation and the true solution on the same graph by setting the B data range to f2..f102. You can also reset the A and B data ranges and set the A data range to G2..G102 to graph the error.

*4 Word processing note: You can save these graphs in .CGM or, in older versions, .PIC files by choosing save in the graph menu. These .CGM or .PIC files can be imported into word processing packages like Word Perfect and incorporated into documents such as this one. Graphs produced by this method are in figures 2 and 3 below.
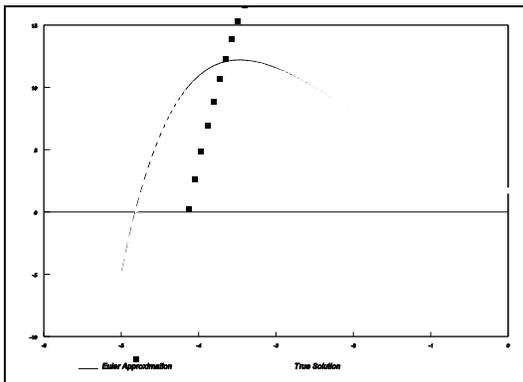

Figure 2: Note that both solutions appear almost to coincide. Some variation appears on the screen in color.
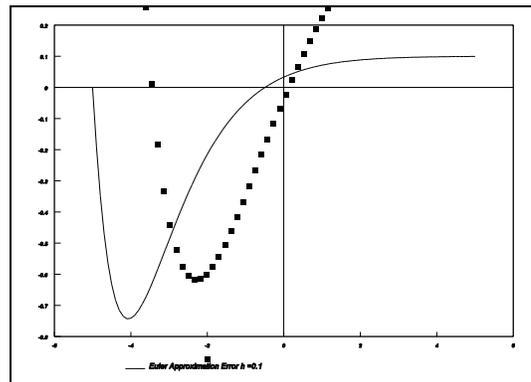

Figure 3: Error in Euler's Method The vertical axis shows number of thousandths

*5   To change h, type the new value for h in cell E2. If you press the **F10** key you will go directly to the graph. Notice that the x value range has changed. If you wish to see a different range, repeat the steps in *4 to reset the X and A and B data ranges.

*6   To Change differential equations to y' = f(x,y): Go to cell B2. Type the formula for the function f(x,y) using B2 and C2 as arguments (instead of f(x,y) type f(B2,C2)). The formula must begin with a number or an operand, like +. If there isn't one already start the formula with +. Follow the copy commands in step 3 to copy this formula to the rest of the column. To change the initial conditions, type the new x_0 and y_0 values in cells B2 and C2. Choose a value for h and look at your graph. To find about how to use library functions like @sin or @exp in f(x,y) use the Help feature by pressing the **F1** key, highlighting **Function Index**, pressing <**enter**> and cursoring through the function index.

The figure below illustrates the formulae needed for Runga Kutta in a manner similar to that for Euler's Method in Figure 1 above.

	A	B	C	D
1	n	xn	yn	k1
2	0	-5	-5	+$H$2*(B2^2-C2)
3	+A2+1	+$B$2+$H$2*A3	+C2+(D2+2*E2+2*F2+G2)/6	+$H$2*(B3^2-C3)
4	+A3+1	+$B$2+$H$2*A4	+C3+(D3+2*E3+2*F3+G3)/6	+$H$2*(B4^2-C4)
5	+A4+1	+$B$2+$H$2*A5	+C4+(D4+2*E4+2*F4+G4)/6	+$H$2*(B5^2-C5)
6	+A5+1	+$B$2+$H$2*A6	+C5+(D5+2*E5+2*F5+G5)/6	+$H$2*(B6^2-C6)

Figure 4a: Lotus Formulae for Classical 4th Order Runga Kutta on y'= x^2 - y; y(-5) = -5

E	F	G	H
k2	k3	k4	h
+$H$2*((B2+$H$4)^2-(C2+D2/2))	+$H$2*((B2+$H$4/2)^2-(C2+E2/2))	+$H$2*(B3^2-(C2+F2))	0.5
+$H$2*((B3+$H$4)^2-(C3+D3/2))	+$H$2*((B3+$H$4/2)^2-(C3+E3/2))	+$H$2*(B4^2-(C3+F3))	h/2
+$H$2*((B4+$H$4)^2-(C4+D4/2))	+$H$2*((B4+$H$4/2)^2-(C4+E4/2))	+$H$2*(B5^2-(C4+F4))	+H2/2
+$H$2*((B5+$H$4)^2-(C5+D5/2))	+$H$2*((B5+$H$4/2)^2-(C5+E5/2))	+$H$2*(B6^2-(C5+F5))	
+$H$2*((B6+$H$4)^2-(C6+D6/2))	+$H$2*((B6+$H$4/2)^2-(C6+E6/2))	+$H$2*(B7^2-(C6+F6))	

Figure 4b: Lotus Formulae for Classical Fourth Order Runga Kutta Method, continued.

These directions have been prepared so that they can be used with a minimum of overhead and transformed for use on other spreadsheets with a minimal knowledge of the particular package; LOTUS and other spreadsheets have ways to define functions that can, for example, decrease the number of changes necessary when changing differential equations. You may wish to explore the special features of your package if you will be doing many of these.
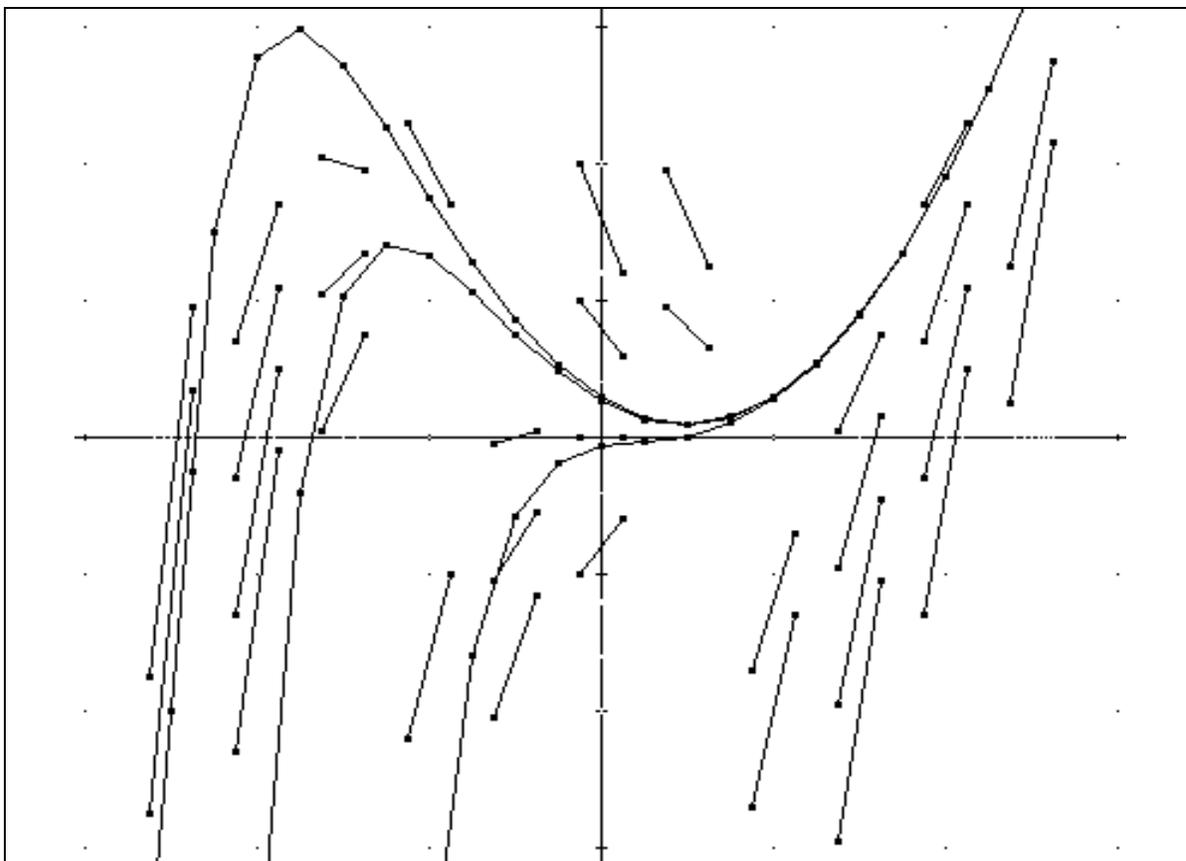
Using the Computer Algebra Package DERIVE

Figure 5 Derive generated direction field and Euler approximations for y'=x^2 - y

Standard computer algebra packages like DERIVE also can be used to approximate solutions

to Differential Equations. In fact, since many Calculus texts and syllabi include a discussion
of differential equations it is not difficult to talk about and use Euler's method and direction
fields in these classes as well. Many calculus classes use technology regularly which makes
incorporating these routines a natural thing to do. My Calculus II students find direction fields
in DERIVE a very useful conceptual tool and a good break from computing integrals. They can
all sketch approximate solutions to a Differential Equation and seem to have a better
understanding of what these equations mean.

There are Utility File Functions in DERIVE that are essentially programs to do more
sophisticated calculations that the standard menu allows. Chapter 9 of these utilities consists
of packages for solving and approximating solutions to differential equations. File 9.8 contains
a program to create Direction Fields and programs for Euler's Method, Runga Kutta and other
approximation methods. Directions for these can be found using the on-line help function in
DERIVE. Direction fields and Euler's method "solutions" to y' = x^2 - y for different initial
conditions generated in DERIVE using the Utility File Functions found in 9.8
ODE_APPR.MTH are in figure 5. The functions create expressions containing points which
you can plot using the plot commands. You can also see the points in the expression. For
example one of the solutions graphed below contains the points: [[-5, -10], [-4.5, 7.5], [-4,

13.875], [-3.5, 14.9375], [-3, 13.59375], [-2.5, 11.296875], [-2, 8.7734375], [-1.5, 6.38671875], [-1, 4.318359375], [-0.5, 2.6591796875], [0, 1.45458984375], [0.5, 0.727294921875], [1, 0.4886474609375], [1.5, 0.74432373046875], [2, ~1.497161865234375], [2.5, 47938/17441], [3, 22195/4933], [3.5, 33296/4933], [4, 26780/~2819], [4.5, 35942/2819], [5, 16.5]] which can be read in the algebra screen in Derive.

It is easier to use the programs in DERIVE to calculate Runga Kutta and Euler's approximations than it is to program a spreadsheet to do them. However, in the process of programming the spreadsheet students can get a better understanding of the method. Also, using a spreadsheet one can program any explicit finite difference including predictor- corrector methods where the implicit method is iterated a fixed number of times. Using DERIVE requires use of one of the methods in the Utility files. On the other hand, DERIVE as an algebra package also has utilities to solve differential equations analytically. This a spreadsheet is not set up to do. The choice to use DERIVE or spreadsheets or both must depend on the availability of software, the time you have to spend, your clientele and the goals of your course, the logistics of using computers on your campus, and instructor or department philosophies. Both packages are powerful tools and ideally students should learn to use both.

In addition to using DERIVE utility file functions, DERIVE can be used as mentioned in the introduction, to convince students that functions defined as definite integrals are really functions. The last time I taught differential equations, I assigned problem 27 from section 4.10 in Nagle and Saff's *Fundamentals of Differential Equations*. The problem reads as follows:

> Express the solution to the initial value problem y" - y = 1/x; y(1)=0, y'(1) = -2
> Using definite integrals. Using Simpson's rule to approximate the integrals, find
> an approximation for y(2).

Although students were able to use the method of variation of parameters to find the solution:

$$(\#)\quad y(x) = e^{1-x} - e^{x-1} + \frac{e^x}{2} \int_1^x \frac{e^{-t}}{t}\, dt - \frac{e^{-x}}{2} \int_1^x \frac{e^t}{t}\, dt \, ,$$

they did not believe this was a function. After reiterating all the standard mathematical arguments to no avail, it finally occurred to me to ask DERIVE to plot the function. The student reaction was amazing. as soon as they saw the graph in figure 6 on the screen, they believed. I also used DERIVE to calculate, effortlessly, a few values, but in this case a picture was worth a thousand numbers.
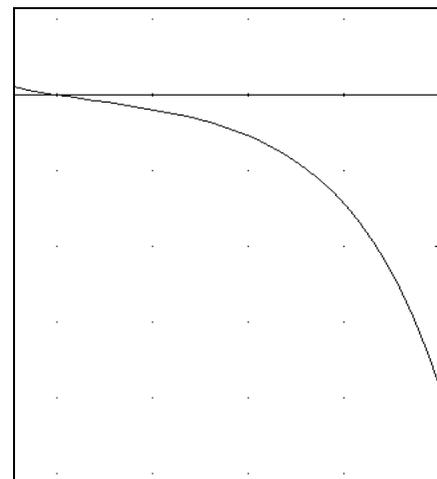

Figure 6: Graph of (#)
From DERIVE

Many students also found DERIVE an essential tool for relieving themselves of the tedious algebra involve in checking solutions. It is easy to type in a candidate for a solution to a differential equation, have DERIVE compute its derivatives and use the edit keys to combine these values to build both sides of the equation, simplify and check to see if they

are equal. Certainly this type of tedious calculation is not something on which one wishes to spend class time. With DERIVE in the classroom a professor can satisfy the student's desire for verification of results and respond to questions about homework without sacrificing half a class period to each question.

In conclusion, the ready availability of computer algebra and spreadsheet packages has definitely changed the way I teach differential equations and it is a change for the better. Classes can be more interactive. It is possible to concentrate on the concepts and not get bogged down in calculations. Students frequently get intrigued or excited by the technology. The incorporation of technology in the classroom has definitely made the course one which better prepares students to be users of mathematics in the real world while not detracting from the study of the discipline for its own sake.

Works Cited

[1]   Nagle, R. Kent and Edward B. Saff. Fundamentals of Differential Equations, 3rd ed. Reading, MA: Addison Wesley, 1993.

[2] DERIVE, a Mathematical Assistant, ver.2.52, Honolulu: Soft Warehouse, 1988-1992.

[3] LOTUS 1-2-3, Rel. 3.1+, Cambridge, MA: Lotus Development Corporation, 1989.