MOTIVATING MATHEMATICAL UNDERSTANDING
VIA SURPRISES ON THE TI-85

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Technology has become a powerful tool for better understanding mathematics, and there are many strong arguments for using it in our mathematics classrooms. Graphing calculators or symbolic computer programs allow for

- more realistic applications
- the analysis of real data
- a better understanding of functions
- more student involvement and discovery
- multiple solution approaches to a problem

In this talk we will discuss another way technology can be used to motivate student interest in the classroom. We will present a series of examples on the TI-85 that produce surprising results. These surprises can be the focus of classroom discussion and motivate further mathematical study.

1. Domains.

Are the functions $y_1 = 2 \ln x$ and $y_2 = \ln x^2$ identical? By graphing them on your graphing utility, you will see that their domains are different.

$$2 \ln x = \ln x^2 \quad \text{for } x > 0$$

The familiar properties of logarithms, such as $\ln a^b = b \ln a$ and $\ln(ab) = \ln a + \ln b$ must be applied with care.

2. Complex Numbers.

Graph the functions $y_1 = \ln x$ and $y_2 = |\ln x|$ (abs ln x). Why does the graph of the second function include points to the left of the $y-$axis? Although the logarithm of a negative number is complex, its absolute value is real, and the calculator plots the point. Note that the TI-85 (and the HP 48G) evaluate functions in the complex domain, unlike the TI-81 and TI-82. Hence, you can get some surprising results in

the classroom. Try evaluating the following expressions on the TI-85.

$$\ln(-2)$$
$$\sqrt{-4}$$
$$\arcsin 3$$
$$e^{i\pi}$$

3. Periodic Functions.

Graph the function $y_1 = \cos 48x$ on the trigonometric window (ZTRIG). Can you deduce the period of the function from its graph? Try tracing along the curve to see what has happened. The TI-85 has 126 horizontal pixels, and for the ZTRIG window, $\Delta x = \pi/24$. Hence, for any pixel x, $\cos 48x = 1$. You might find it interesting to produce other surprising graphs of trigonometric functions, such as $\sin 48x$, $\sin 49x$ and $\cos 23x$.

4. Inverse Functions.

The functions $f(x) = \sin x$ and $g(x) = \arcsin x$ are inverses of each other. Does that mean that their composition is the identity function? The graphs of $y_1 = \sin(\arcsin x)$ and $y_2 = \arcsin(\sin x)$ tend to surprise my students. I always make a point in class that graphing utilities force us to think about domains much more than before. For example, what happens when you calculate $\sin(\arcsin 2)$ and $\arcsin(\sin 3\pi)$?

5. Roundoff Error.

You should always be aware that the algorithms used by calculators only produce approximations, and that roundoff error can sometimes yield a poor result. For example, calculate $64^{1/3}$ and compare it to $[[64^{1/3}]]$ (int $64 \wedge (1/3)$). Why are the answers different? You'll see why if you calculate $64^{1/3} - 4$.

The same phenomenom occurs in matrix computations. Enter the matrix

$$\begin{bmatrix} 3 & 11 \\ 2 & 6 \end{bmatrix}$$

and calculate both its determinant, det A, and the greatest integer of the determinant, int det A.

6. Why Can't I Graph $f(x) = x^{2/3}$.

This is one of the most common questions asked about graphing calculators! If you try to graph $f(x) = x^{2/3}$ as $y_1 = x \wedge (2/3)$ you will only obtain the right hand branch of the graph. However, $g(x) = x^{1/3}$ ($y_2 = x \wedge (1/3)$) produces the correct graph. The reason depends on the design of the calculator, and has been discussed frequently on the various electronic bulletin boards. It should be noted that you can obtain the correct graph of $f(x) = x^{2/3}$ using $y_3 = (x \wedge (1/3))^2$ or $y_4 = (x^2) \wedge (1/3)$.

You might find it surprising to graph the derivatives of $f(x) = x^{2/3}$ and $g(x) = x^{1/3}$ on the viewing rectangle $[-3, 3] \times [-2, 2]$ using both the nDer command (numerical derivative) and der1 (exact derivative).

$$\text{nDer}(x \wedge (1/3),x,x)$$

$$\text{der1}(x \wedge (1/3),x,x)$$

$$\text{nDer}(x \wedge (2/3),x,x)$$

$$\text{der1}(x \wedge (2/3),x,x)$$

7. More Derivatives.

What is the derivative of $f(x) = |x|$ at $x = 0$? If you type in nDer(abs x,x,0), you'll see a surprising answer. (Make sure your tolerances are set to 10^{-4}) The TI-85 uses the numerical approximation

$$f'(x) \approx \frac{f(x + \delta) - f(x - \delta)}{2\delta}$$

which explains the apparent "mistake". Other calculators, such as the Sharp EL-9300, use the more familiar forward difference formula,

$$f'(x) \approx \frac{f(x + \delta) - f(x)}{\delta}.$$

The nDer command can lead to other surprising results. Try estimating the derivative of $f(x) = 1/x$ at $x = 0$.

8. More Calculus.

You have to be especially careful when using a graphing utility in calculus. For example, try checking to see if $y_1 = x^4 - 3$ has any inflection points. Or, integrate $f(x) = 1/x$ on the interval $[-1, 1.233]$,

$$\text{fnInt}(1/x, x, -1, 1.233)$$

9. Parametric Equations.

Graphing calculators are particularly useful for graphing polar curves and parametric curves. However, it is important to distinguish between the graph of a curve given by a set of parametric equations, and the equations themselves. Consider the following two sets of parametric equations.

$$\text{xt1} = \cos t$$
$$\text{yt1} = \sin t$$

$$\text{xt2} = \cos t^2$$
$$\text{yt2} = \sin t^2$$

defined on the $t-$domain $-\pi \leq t \leq \pi$. Are the graphs of these equations the same? Are these parametric curves the same? Look closely as your graphing utility generates the second curve. For instance, is the second set of equations smooth at the point $t = 0$? You might find it useful to trace along the graph near $(1, 0)$.

Conclusion.

I hope you have found some of these examples amusing and instructive. I am sure you have discovered your own "surprises" while using graphing technology in the classroom. Rather than becoming worried or upset when the calculator behaves unexpectedly, I believe we should welcome the surprise and use it to better understand the underlying mathematics.