Nonlinear Differential Equations in *Mathematica* ™

by
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Second and third year courses in Advanced Engineering Mathematics or differential equations traditionally cover models or equations that admit a simple closed form solution. Techniques such as perturbation and group analysis, while in principle no more difficult than some of the more traditional topics, cannot at present be effectively taught at this level because the intermediate computations are so complicated. The advent of *Mathematica* ™ and other such systems changes this. We give an example of how *Mathematica* ™ can be used in a classroom situation to study nonlinear ordinary differential equations. In particular, we show how *Mathematica* ™ can be used to study periodic solutions to the Duffing equation by means of perturbation .

In various courses, entitled Advanced Engineering Mathematics or differential equations various classical ordinary differential equations are studied. These include first order equations, second order linear equations with constant coefficients, and if time permits, some particular second order linear equations such as Bessel's equation. Increasingly, such courses include some numerical techniques for finding approximate solutions to those equations that one cannot solve explicitly. One such equation is Duffing's equation, which models a hard spring:

$$y'' + c^2y + \mu hy^3 = 0,$$

subject to the initial conditions $y(0) = A$, $y'(0) = 0$. This equation models a hard spring, i.e., where the restoring force $c^2x + \mu hy^3$ of the spring depends nonlinearly on the displacement y. When $\mu = 0$, this is just an ordinary linear spring. By making a change of variables, (and changing μ and A) we may rewrite this equation (with initial conditions) as

$$x'' + x + \mu x^3 = 0 \qquad (1)$$
$$x(0) = A \;,\; x'(0) = 0$$

Where x(t) is the (scalled) displacement. If $\mu = 0$, then it is well known that the solution is A cos(t). (Guess $C_1 \cos(t) + C_2 \sin(t)$. Given that $x(0) = A$ and $x'(0) = 0$, then the solution is A cos(t).) By numerically solving Duffing's equation (1) for values of μ near zero, we can get some idea of how the nonlinear term " μx^3" contributes to the

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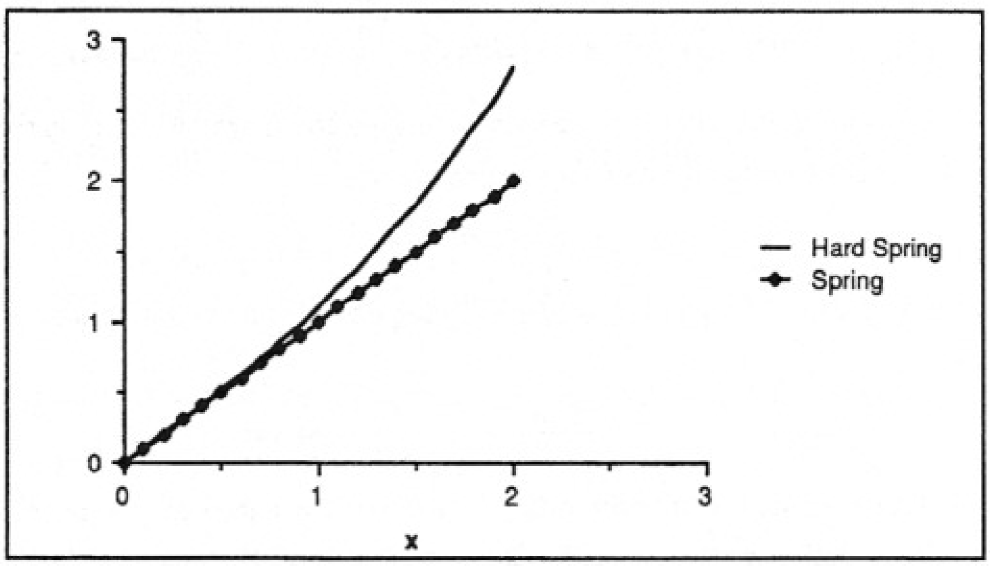
qualitative behavior of the solution. But by doing this we really don't obtain any hard information, all that we get is our impression obtained by looking at several graphs. To obtain more information, we must use some of the analytical techniques for finding approximate solutions. Unfortunately, many of these techniques require extensive computation. However, it is this author's claim that some of these methods are no more difficult in principle than some of the methods currently in the curriculum. In what follows, we study, with the help of *Mathematica* ™, approximate analytic solutions of the the Duffing's equation. In particular, we obtain approximate solutions that are accurate within a given power of μ. This particular problem is chosen for at least two reasons:

1. The computations, while laborious, can be done by hand. When stating off with a new technique with a symbolic manipulator, it is important to be able to relate what is going on inside the computer system to what we already can do. I would not recommend actually carrying all of the details out be hand, but one can do some of the parts by hand.

2. The answer to this problem is (at least in some circles) well known. Thus the instructor will either know of this example, or can look it up in many standard texts, e.g., [1,2].



The restoring force of a spring and a hard spring -- $\mu = 0.1$

All right, here we go. Let N = 1 (or 2 latter on, or larger if you have the patience or computer time). We will look for periodic solutions. Of course the period of the solutions depends on μ. If $\mu = 0$ then solutions are periodic with period 2π. Let us guess that the period as a function of μ is $2\pi\omega$ where:

$$\omega = 1 + \mu\omega_1 + \mu^2\omega_2 + \mu^3\omega_3 + \ldots + \mu^N\omega_N + O(\mu^{N+1}) \qquad (2)$$

We will guess that x(t) can be written as a series on μ:

$$x(t) = u_0(\omega t) + \mu u_1(\omega t) + \ldots + \mu^N u_N(\omega t) + O(\mu^{N+1}) \tag{3}$$

Where each $u_i(\cdot)$ is a periodic function of period 2π. Letting $\tau = \omega t$, then Duffing's equation (1) becomes:

$$\omega^2 \frac{d^2x}{d\tau^2} + x + \mu x^3 = 0 \tag{4}$$

$$x(0) = 0 \quad , \quad x'(0) = 0$$

Using *Mathematica* ™ (or doing the computations by hand) we plug (2) and (3) into (4), keeping only the terms of order less than or equal to μ^N. We now collect terms based on powers of μ. The zeroth order terms give:

$$\mu^0: \quad \frac{d^2u_0}{d\tau^2} + u_0 = 0.$$

This is easily solved, using the initial conditions $u_0(0) = A$, $u_0'(0) = 0$, to give $u_0(\tau) = A\cos[\tau]$. This now tell us that:

$$x(t) = A\cos(\omega t) + O(\mu), \text{ where } \omega = 1 + O(\mu).$$

Plugging back this value for u_0 into the expanded equation, we get the first order terms:

$$\mu^1: \quad \frac{d^2u_1}{d\tau^2} + u_1 = \left(2\omega_1 A - \frac{3A^3}{4}\right)\cos(\tau) - \frac{A^3}{4}\cos(3\tau)$$

The solutions of this will not be periodic unless the coefficient of $\cos(\tau)$ on the right hand side is zero. So we set this coefficient equal to zero:

$$\left(2\omega_1 A - \frac{3A^3}{4}\right) = 0$$

and solve for ω_1. This gives $\omega_1 = \frac{3a^3}{8}$. Plugging this back into the μ^1 equation above, we get:

$$\frac{d^2u_1}{d\tau^2} + u_1 = -\frac{A^3}{4}\cos(3\tau)$$

If we find a solution of this with $u_1(0) = 0$ and $u_1'(0) = 0$, then $u_0(\tau) + \mu u_1(\tau)$ will satisfy the initial conditions $x(0) = A$, $x'(0) = 0$. We do this (in *Mathematica* ™) by the method of undetermined coefficients, and conclude that

$$u_1(\omega t) = \left(\frac{-A^3\cos[\omega t]}{32} + \frac{A^3\cos[3\omega t]}{32}\right),$$

where $\omega = 1 + \frac{3a^3}{8} + O(\mu^2)$. Remembering back to our guess for x, this gives

$$x(t) = A\cos[\omega t] + \mu\left(\frac{-A^3\cos[\omega t]}{32} + \frac{A^3\cos[3\omega t]}{32}\right) + O(\mu^2)$$

Note that this (approximate) solution of Duffing's equations tells us more than we could have obtained from studying plots of numerical solutions. It in fact tells us what the (approximate) frequency shift from the harmonic oscillator (μ=0) case, and the presence of the "harmonic component" cos(3ωt). Carrying out the second order calculation in *Mathematica* ™ we get:

$$x[t] = A\cos[\omega t] + \mu\left(\frac{-A^3\cos[\omega t]}{32} + \frac{A^3\cos[3\omega t]}{32}\right)$$
$$+ \mu^2\left(\frac{23\,A^5\,\cos[\omega t]}{1024} - \frac{3A^5\cos[3\omega t]}{128} + \frac{A^5\cos[5\omega t]}{1024}\right) + O[\mu^3]$$

where

$$\omega = 1 + \frac{3\,A^2\mu}{8} + \frac{21\,A^4\,\mu^2}{256} + O[\mu^3]$$

Note: The technical report version of this paper contains an appendix listing the *Mathematica* ™ notebook **Duffing**, in which the computations of this paper are done. This may be obtained from the author.

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