Using Computers to Teach Series and Differential Equations

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At Grinnell College we teach "Series and Differential Equations" to our second semester sophomores. In this course we study notions of convergence and approximation — topics that are inherently geometric. Traditionally these topics are taught with the aid of rough sketches on a blackboard. Some beautiful but static graphs appear in the texts. But students — especially those who are weak on the concepts — cannot produce correct sketches on their own. Computers have been used to teach isolated topics. But the computer programs have not generally been available for student use.

Computers, with their ability to perform numerical calculations and produce high-resolution graphics, can be used to present ideas of convergence and approximation. The problem is to choose software that is both powerful and easy to use. A general graphics package will graph any function you choose. But to graph, say, the partial sum of a Fourier series would require entering a very ugly function. On the other hand, special-purpose educational software often limits the functions that can be graphed. In some instances, the canned examples are little better than static drawings.

In the fall of 1987, I started work on a project to use the computer to teach concepts of convergence and approximation. My main goal was to provide an environment in which the student could experiment with a wide variety of functions. I wished to use the computer to enhance intuition through graphics, and to do some of the otherwise tedious or impossible numeric computations. The computer software must be so **easy to use** that attention is focused on the mathematical concept and not on the computer. It must provide students with as much **control** as possible over the output, yet make controls optional so reasonable graphs may be produced with little effort. And it must be accompanied by examples and exercises that encourage the student to explore examples, make conjectures, and back up intuition with rigorous mathematical argument.

I wrote and used early versions of some of the software and exercises in the spring of 1988. In the summer the Sloan Foundation supported work on improving the modules, implementing new modules, and writing supporting materials. These are being used this fall. Modules have been written to graph functions of one and two variables, to graph sequences of numbers (x_n vs. n), sequences of functions ($f_n(x)$), Taylor polynomial approximations to functions, partial sums of power series and Fourier series, and numerical solutions to equations, definite integrals, and first- and second-order differential equations. The computer modules are written in Matlab, a software package that provides matrix operations graphics calls. The software runs at Grinnell College on a network of Sun 3/50's. Computer monitors in classrooms allow use of the software for class demonstration. Students use the laboratory equipped with 18 workstations to run the software, producing graphs and numeric output on the screen and printing graphs and tables on the laser printer.

To illustrate how the software is used, I have chosen to focus on seven mathematical problems.

1. **Uniform convergence of functions:** Figure 1 shows a graph of the first 7 functions in the sequence $f_n(x) = nx \cdot e^{-nx}$. From the graph students can conjecture that each f_n has one critical point, and the maximum values for the functions appear equal. Calculus shows this critical point to be at $x = 1/n$, with $f_n(1/n) = 1/e$. The functions do not converge uniformly to 0 on $[0, \infty)$. But for $c > 0$, convergence is uniform on $[c, \infty)$. From the graph it is clear we must argue that past a point in the sequence f_n is decreasing on $[c, \infty)$. Graphing leads to conjectures which then must be proved.

2. **Taylor's Remainder Formula:** The sixth Taylor polynomial for $\sin(x)$ evaluated at x^2 is: $P_6(x^2) = x^2 - x^6/3! + x^{10}/5!$. Using Taylor's Remainder Formula, we find $|\sin(x^2) - P_6(x^2)| \leq$

$(x^2)^7/7! \quad \cong .744$ at $x = 1.8$. The graph in Figure 2 illustrates the actual error at 1.8. Further, if we wish to approximate $\int_0^{1.8} \sin(x^2)\,dx$ by $\int_0^{1.8} P_6(x^2)\,dx$, the actual error is represented by the area between the two functions. Certainly an upper bound for that error is 1.8 times the maximum error. But a much tighter upper bound is the integral of the TRF bound, $\int_0^{1.8} x^{14}/7!\,dx$.

3. **Interval of Convergence:** In Figure 3 we graph the partial sum of the Maclaurin series for $f(x) = \ln(1+x)$ for $n = 3, 6, 9, 12$. The graph suggests the partial sums converge to the function for some values of x, but diverge for $x > 1$. Intuition motivates a more careful study.

4. **General solution to a first order differential equation:** In Figure 4 we have graphed the direction field for the differential equation $y' = 1 - 2xy$. Initial values for solutions were chosen by moving the mouse to a point (x_0, y_0) and clicking. From this simple graph the student gains an intuitive notion of the family of curves that make up the general solution to the differential equation. Also, clicking the mouse in the general vicinity of (2,0) shows the extreme sensitivity to initial values in some regions.

5. **Numerical solutions to first order differential equations:** In Figure 5 the solid line represents the solution to the initial value problem $y' = 3yx^2 \cdot \cos(x^3), \quad y(0) = 1$. The dashed line is the numerical approximation using Euler's method with $n = 100$ intervals. The graph vividly shows that the extreme curvature of the function makes it difficult for Euler's method to track. Euler's improved method and the Runge-Kutta method of order 4 do much better. The graphs in Figure 6 show the error in the Euler approximation with $n = 2000$ and the difference between the Euler approximations at $n = 2000$ and $n = 1000$. The similarity of the graphs helps to support the rule of thumb that the error at $2n$ intervals approximately equals the difference between approximations at $2n$ and n intervals, thus providing a stopping criterion.

6. **Series solutions to second order differential equations:** Computing series solutions to second order differential equations is a messy business. The power series solution to the differential equation $y'' = -xy' - y$ with initial values $y(0) = 1$, $y'(0) = 2$ is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k \cdot k!} \cdot x^{2k} + \frac{2 \cdot (-1)^k}{1 \cdot 3 \cdots (2k+1)} \cdot x^{2k+1}$$

The behavior of this function is far from obvious. In Figure 7 we have graphed a numerical solution to the initial value problem together with partial sums of the series solution. The partial sum S_{49} is indistinguishable from the numerical solution. This computer tool not only gives the students a way to check their power series solution. It also provides an approximate graph of the function.

7. **Fourier series:** It is important to support the idea of adding terms in a Fourier series with graphical results. The module to graph partial sums of Fourier series allows the user to specify the period of the function $f(x)$ and to define the function over that period piecewise. In Figure 8 we have graphed the function that is periodic of period 2π, and is defined to be 0 on $[-\pi, 0)$ and x on $[0, \pi]$. Once the formulae for the Fourier coefficients have been entered, the student can graph the partial sum of the Fourier series for any reasonable n. The Gibbs phenomenon is observed at points of discontinuity. It is also enlightening to graph the difference between f and the partial sum.

Conclusion: Educational software can give students insight and power to solve a much wider range of problems than previously possible. The software must be both flexible and easy to use. Its success must be measured not only by its ability to demonstrate principles, but by its ability to put useful tools in the hands of the students.

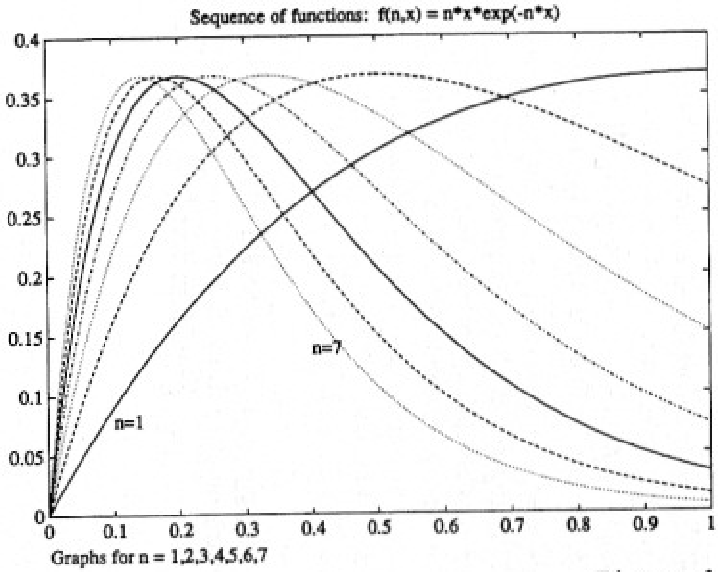

Figure 1

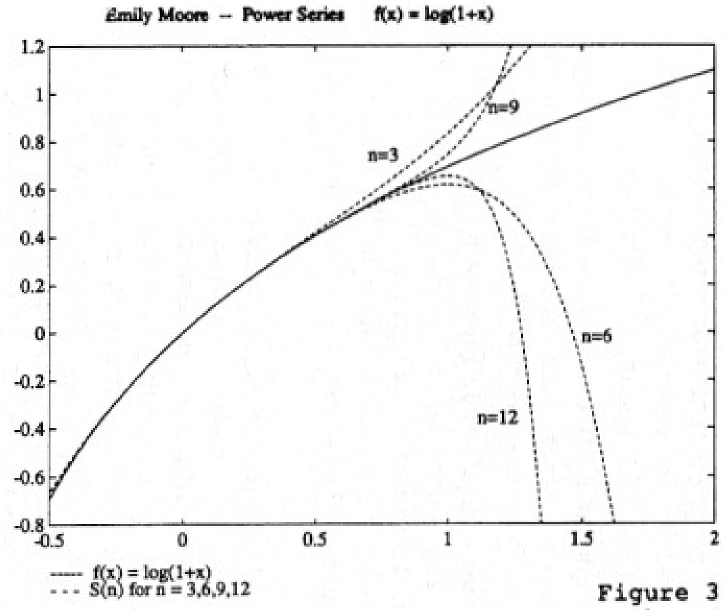

Figure 3

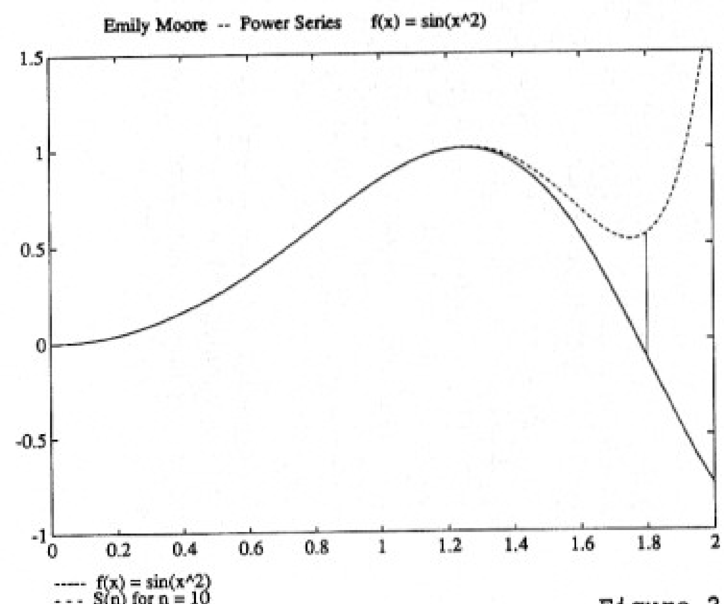

Figure 2

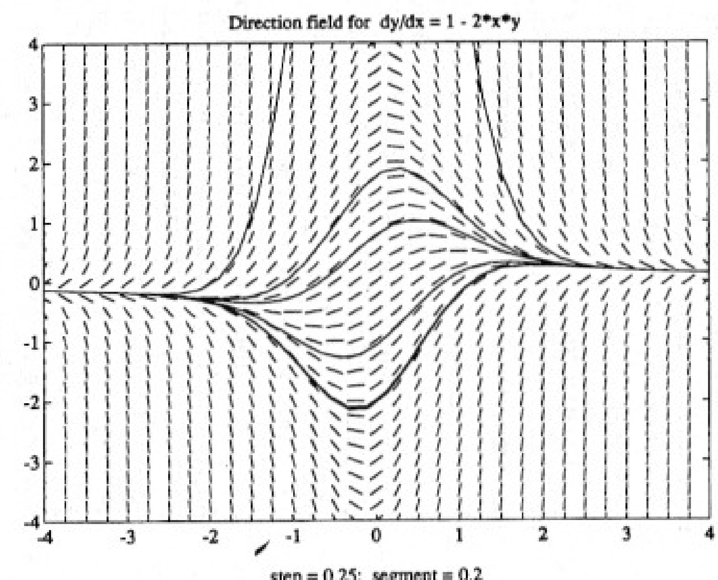

Figure 4

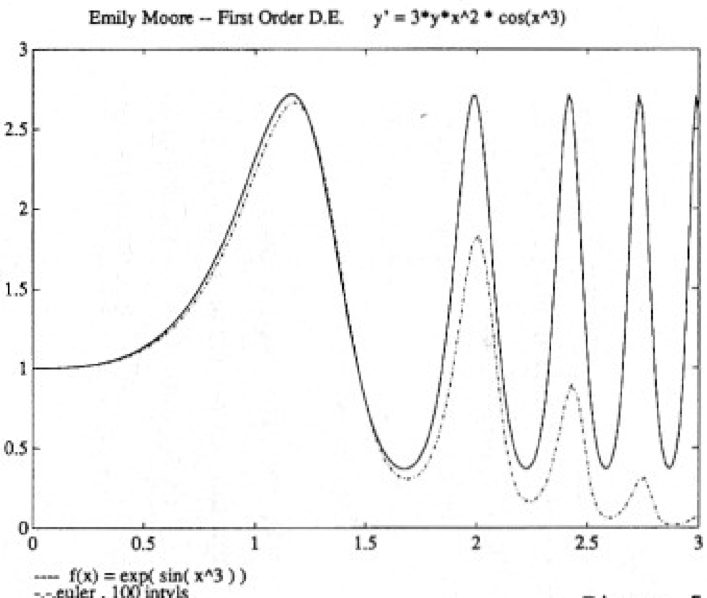

Figure 5

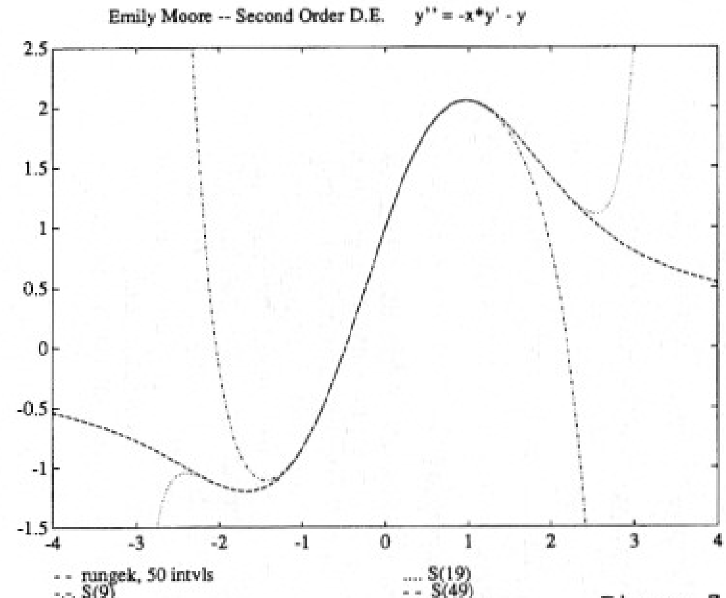

Figure 7

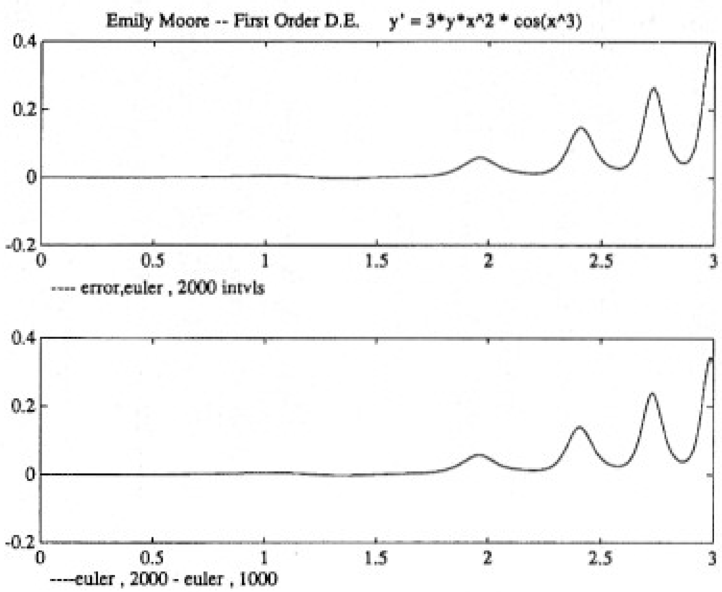

Figure 6

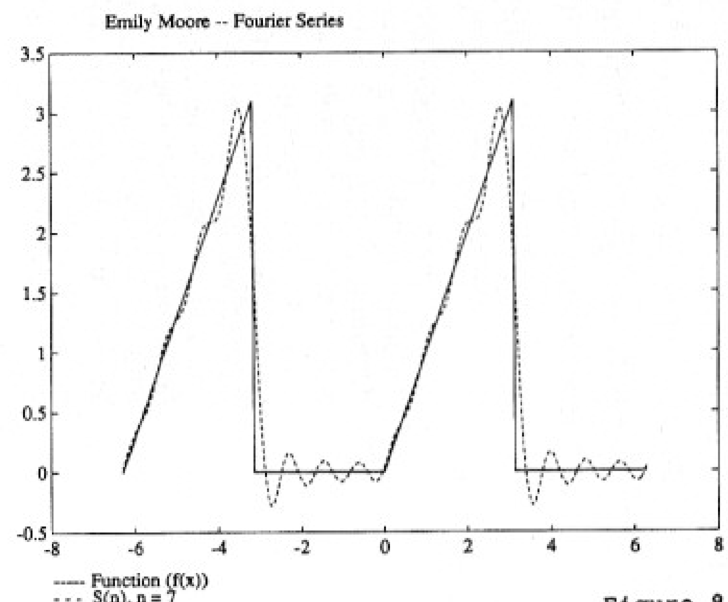

Figure 8