Algebraic, Graphical, and Numerical Computing in Elementary Calculus: Report of a Project at St. Olaf College[1]

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Introduction

Computer algebra systems (hereafter, CAS's) offer important new possibilities for mathematical teaching and learning. CAS's provide, for the first time, a powerful and varied range of mathematical computing tools. Using these tools, students can study and represent mathematical ideas more effectively, more efficiently, and more flexibly than before. Just as important, these tools come in a single, easy-to-use package, at an acceptable cost in time and distraction from mathematics itself.

In this paper I describe a CAS-oriented project at St. Olaf College. The project aims to bring algebraic, numerical and graphical computing power to bear on the teaching and learning of elementary calculus.

Educational computing in general; CAS's in particular

Much has been said and written recently (see, e.g., [1], [2], [4], [5], [8]) about how computing can help improve undergraduate mathematics. Among the candidates for improvement, elementary calculus courses are clear contenders. Among the proposed improvers, CAS's are frequently singled out, for two main reasons. First, CAS's do more than other programs; second, they do what they do more easily. (For more information on CAS's, see, e.g., [3], [6], [7], and parts of [2] and [5].)

A few background observations on CAS's will put things in context.

- The term "CAS", although standard, is misleading; CAS's do much more than algebra. They are better seen as mathematical toolkits, with graphical and numerical, as well as symbol-manipulating capabilities.
- CAS's can operate *without programming*, in "calculator mode"; a one-line input command is usually enough. (On the other hand, a CAS *can* be regarded as a high-level mathematical programming language, in which mathematical algorithms are easily realized. This raises its own pedagogical possibilities, but I do not pursue them here.)
- Elementary calculus students can easily learn to use a CAS effectively at an appropriate level; no great mathematical sophistication is needed. (Obviously using a powerful CAS to its *full* mathematical potential which goes far beyond elementary calculus, requires commensurate mathematical knowledge.)

Calculus students at St. Olaf College use SMP, typical of powerful computer algebra systems. A few local features were added, such as a simple help system and several comands for numerical methods. For example, the Midpoint command used below was created locally. It is easy to create such local variations for special pedagogical purposes.

Figure 1 gives a brief SMP session. Commands are chosen to illustrate the combination of algebraic, graphical, and numerical capabilities, and to show how easily the power of the CAS is tapped. Input lines (the ones typed by the user) begin with # I[ ]::. Outputs, marked with #O[ ]:, are returned by the machine. The meaning of most of the lines would be clear from context: D denotes *derivative*; Int means *integral*; Midpoint refers to the midpoint approximation to a definite integral; input command 6 forces a decimal version of the previous line.

```
# I[1]:: D[Sin[x],x]
# O[1]:  Cos[x]

# I[2]:: Graph  [Sin[x], x, 0, 1]
# O[2]:  < a graph appears; units are shown on
           both axes >
# I[3]:: Int[Sin[x],x]
# O[3]:  - Cos[x]

# I[4]:: Int[Sin[x], { x,0,Pi } ]
# O[4]:  2

# I[5]:: Midpoint[Sin x,x,0,Pi,6]
# O[5]:
```

$$\frac{\text{Pi}(\text{Sqrt}[2] + \text{Sin}\frac{\text{Pi}}{12} + \text{Sin}\frac{5\text{Pi}}{12} + \text{Sin}\frac{7\text{Pi}}{12}\text{Sin}\frac{11\text{Pi}}{12})}{6}$$

```
# I[6]:: N[%]
# O[6]:  2.02303
```

Figure 1. SMP Session

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Specifics of the St. Olaf College CAS project

The St. Olaf College mathematics department is carrying out a three-year project to incorporate symbolic, numerical, and graphical computing into elementary calculus courses. Support has been provided by the National Science Foundation and the Fund for Improvement of Postsecondary Education (FIPSE).

Four special sections of elementary calculus, enrolling about 120 students, are taught each semester. Students in these sections use the computer algebra system SMP, to complete specially-prepared homework assignments and, sometimes, in taking examinations. A network of ten powerful, graphics-capable SUN workstations, together with additional remote terminals, is provided for project use.

The syllabus for project sections includes (at least) the nucleus of the usual course. A few non-standard topics, such as numerical methods for integration and root-finding, are covered. Perhaps the most distinctive properities of the special sections, though, are their general viewpoints: to emphasize ideas over mechanics (though not to neglect the latter), and, to draw freely on geometric, numerical, and algebraic methods for learning, illustrating, and applying ideas. Thus, for example, while traditional courses often emphasize graph-*sketching*, project sections freely use computer graphics to illustrate and emphasize connections between analytic properties of a function and geometric properties of its graph.

Students' special experience in project sections stems mainly from working special homework exercises and tests, designed to exploit SMP's various computational powers. Writing and improving these course materials is a major faculty effort. The purpose of the materials is at least twofold. First, they help students use calculus ideas and techniques more powerfully than usual (as, e.g., in combining exact and approximate techniques for integration). Second, and just as important, they can be used to illustrate and teach calculus ideas more clearly in the first place.

Philosophy of the St. Olaf project

It is to be emphasized that the St. Olaf project does *not* aim to "automate" a standard elementary calculus course. Perhaps, some routine calculus manipulations *should* be mechanized, but doing *only* that would be worse than useless. The principal goals of the St. Olaf College project are conceptual: to help students understand calculus ideas more deeply and to apply them more effectively.

The special opportunity CAS's offer is to supplement the usual narrowly algebraic approach to the calculus with numerical and graphical viewpoints. In standard calculus courses, key concepts are represented and manipulated almost always in *algebraic* form. Thus, for example, students usually treat limits, derivatives, and integrals–all *analytic* objects–only as *algebraic* operations on *algebraic* functions. When such closed-form methods fail, students have little recourse. Such a circumscribed view of important ideas leads to a rigid, shallow, mechanical calculus course, ill-suited to convincing applications.

CAS's allow a variety of "representations" of mathematical ideas. Numerical and geometric viewpoints that traditionally require forbiddingly tedious and repetitive operations are possible with a powerful CAS.

A calculus example using SMP

Suppose $g(t) = \sin(t^2)$ and $G(x) = \int_0^x g(t)dt$.

a. Consider the problem of determining the value of $G(1)$. Notice the fundamental theorem does not help, because g has no elementary antiderivative. Estimate $G(1)$ in two ways: first geometrically, as area under a graph. How accurate can your estimate be? Next, approximate $G(1)$ using a midpoint sum with 10 subdivisions.

b. In order to obtain a useful graph of $G(x)$ on $[0, 3]$ we need many (approximate!) values of G. Let SMP do the work, as follows. Give the SMP commands:

```
H[$x]:: Midpoint [Sin[t^2],t,0,$x,10]

Graph [H[x],x,0,3]
```

(The first command defines a function H that numerically approximates G. The $\$x$ notation is simply SMP syntax for function definition. The second command sketches a graph of H.) Discuss the relationship between f and G, by examining graphs of both functions.

SMP commands for the example

Several SMP input and output lines related to the given problem are given in Figure 2.

```
#I [1]:: Graph [Sin[t^2], { t,0,1 }]
#O [1]:  < a graph appears; I omit it. >

# I[2]:: NMidpoint [Sin[t^2], { t,0,1 } ,10]
# O[2]:  0.309816

# I[3]:: NMidpoint [Sin[t^2], { t,0,2 } ,10]
# O[3]:  0.809254

# I[4]:: NMidpoint [Sin[t^2], { t,0,3 } ,10]
# O[4]:  0.795971

# I[5]:: H[$x]::NMidpoint[Sin[t^2], { t,0,$x } ,10]
# O[5]:  'NMidpoint [Sin[t^2], { t,0,$x } ,10]

# I[6]:: Graph[Sin[x^2],H[x], { x,0,3 }]
# O[6]:  < a graph appears; see Figure 3. >
```

Figure 2. Related SMP lines

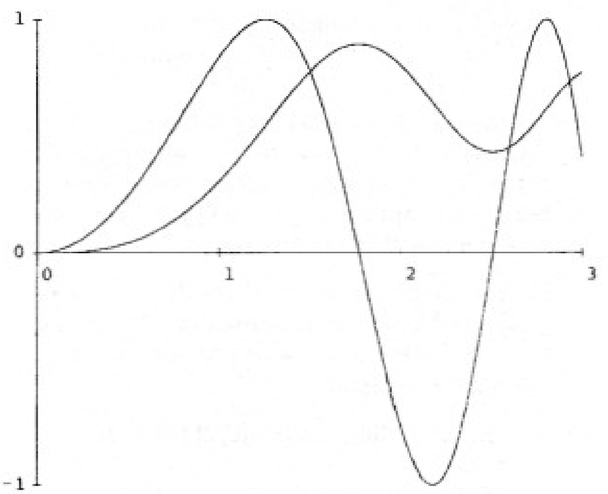

Figure 3. Graphs of $\sin(x^2)$ and H

Remarks on the example

Although the problem uses modern technology, it does so for an entirely traditional purpose: to illustrate and reinforce an important theoretical idea–the connection between derivative and integral. The effect is not to reduce the calculus course to thoughtless button-pushing, but, rather, to shift attention from mechanics to central ideas.

The idea of a function defined by an integral–as opposed to antidifferentiation–is crucial to understanding the fundamental theorem of calculus. Many students miss this important distinction, and therefore see the fundamental theorem as a tautology: the derivative of an antiderivative is the original function. Here the choice of f forces the crucial distinction to be made.

Problems like this require too many calculations for "manual" methods. With a CAS, the full gamut of algebraic, numerical, and graphical capabilities are used. On the other hand, all calculations carried out by the machine are simple and straightforward. Thus, the machine operates "transparently", rather than as a "black box."

Results of the St. Olaf College project to date

Although the project is not completed, some preliminary results and observations can be mentioned.

- Students seem to perceive the CAS-aided calculus courses as more difficult, but also more valuable, than standard courses.
- Learning SMP (its syntax, etc.) is not a serious obstacle for the average calculus student. On the other hand, not surprisingly, mathematically stronger students also make better use of CAS.
- Many beginning calculus students *expect* routine calculations, not concepts, to be the main focus of their courses. More conceptual courses must contend with such expectations.
- Gauging the success of CAS-aided calculus courses relative to standard courses is important, but also difficult. Common examinations, for example, are impractical when course syllabi differ significantly. Nonetheless, early indications (from St. Olaf College and elsewhere) suggest that CAS's offer significant advantages for mastering conceptual topics and that hand manipulation skills are not sacrificed.

Conclusion

CAS's have existed for many years, but they are just now becoming widely available. Soon even handheld machines (like the *HP-28S*) will be able to do much of what we traditionally teach (and test) in elementary calculus courses. Whether we like it or not, courses must change.

Although powerful CAS's are sometimes seen as threats to traditional mathematical education, they should more properly be seen as offering new opportunities to deepen and revitalize mathematics courses, focusing them more sharply on ideas rather than mechanics.

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