The Twilight of Paper-and-Pencil: Undergraduate Mathematics at the End of the Century

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In the final decade of this century, paper and pencil will take its last stand in mathematics. The signs are everywhere. When was the last time you multiplied together two 3-digit numbers? Or performed long division? Or computed a square root to two decimal places by hand? Or used a table of logs or cosines? We do it on a calculator or in our heads; we don't do it on paper. In the same way that simple calculators have washed away paper-and-pencil arithmetic (at least for adults), more sophicated calculators and computers will soon wash away pencil-and-paper algebra. The tide is rising, and, like any tide, there will be no way to stop it.

And yet, grade schoolers still are drilled endlessly in long multiplication and division. Some probably are still being taught to take square roots by hand. My tenth grade daughter, Emily, still has to do twenty problems a night "simplifying" $\sqrt{128/3}$ to $(8/3)\sqrt{6}$ (it makes you wonder what's left for punishment when the class gets unruly, as it probably should). High school students are still taught how to use tables of logs. Every calculus book still has tables of trigonometric functions. And perhaps most tellingly, no national mathematics test taken by student under the age of 18 allows calculators at the present time.

Other fields seem perfectly willing to embrace technology. I don't hear my colleagues in physics wailing because calculators now contain all important constants and conversion factors so their students don't know five digits of Avogadro's number anymore. Home economics classes use calculators or mental arithmetic; Emily says she was taught to compute the tip at a restaurant just by doubling the 7 1/2% state tax already on the bill. No English professors have banned word processors on their paper assignments. Calculators are allowed on, for example, the College Board's Advanced Placement Chemistry examination.

Why are mathematicians so slow to adopt technology? There are some reasonable excuses. It is difficult to keep up with developments. At the very time the Content Workshop at the 1986 Tulane calculus conference was recommending the use of calculators with integration and root-finding buttons, Casio was releasing the first graphics calculator and Hewlett-Packard was waiting in the wings with graphing and symbolic manipulation. In 1982, Herb Wilf [3] warned the mathematics community about "The disk with a college education," which gave a micro the symbolic power of a mainframe. Five years later I was warning [2] about "The calculator with a college education," which had much of the power of Wilf's micro. Do we even dare to guess what the next five years will bring?

The logistics of using calculators and computers is also daunting to mathematicians. After all, we became mathematicians because we didn't like balky laboratory equipment. Chalkdust runs in our blood. We deliberately sought the asceticism of paper and pencil. Ask any waiter: our napkins are covered with writing, not food. Even if we do put up with an occasional demonstration in class or a few problems in the textbook marked with a calculator icon, we have no intention of allowing them in the examination room. The logistics are just too complicated.

Paul Zorn [4] has said that maybe the problem is that computers and calculators strike much closer to home for mathematicians. The numerical computations we do in chemistry were never really the point of the lesson; so it's fine if a computer does it for us. But numbers and symbols are the point in mathematics. Or are they? Aren't mathematicians always trying to live down their supposed computing ability? "So you're a mathematician. It must be awful adding up all those numbers." The public believes that computers have made mathematicians obsolete. Isn't it ironic that we reinforce that belief by banning technology from our classrooms? Of course, the more sophisticated public knows mathematicians don't just do arithmetic, they do algebra instead, factoring bigger and bigger polynomials I suppose. Does that mean our obsolescence was only postponed until computers learned to do algebra? I'll be honest. My students are frequently better at algebra than me. In all my years in the AP Calculus program, the hardest problem for me was always that one precalculus problem on the AB exam where you had to factor a cubic–of course the students did great on that one.

I refuse to believe that we became mathematicians because we enjoyed mechanical manipulation so much. Do computers and calculators strike too close to home for mathematicians? I certainly hope not.

No, I suspect the reluctance of mathematicians to use technology is based on a much deeper fear, a fear that something beautiful, pure, and eternal will be lost. Tying oneself to technology is tying oneself to the messy, the experimental, and the present. Somehow, paper and pencil are the tools of civilization, while computers are the tools of barbarians. I think the following benchmark problem does a reasonable job of measuring that fear. It is a multiple choice question. You are, in the best ETS tradition, to circle the "best" answer.

$$\int_0^1 \frac{1}{1+x^2}dx =$$

a) .785398163398	b) .785351335767
c) arctan 1 - arctan 0	d) ln2 - ln1
e) 1 - 1/3 +1/5 - 1/7 . . .	f) $\pi/2$
g) $\pi/4$	h) $'\pi/4'$

Here is an explanation of the answers (the "distractor analysis" as it is known in the trade). Option (a) is the correct answer to 12 significant digits. Option (b) is what my HP28S gives me when I push the integral key with the accuracy set at 0.01. Option (c) is a favorite of calculus students, which will turn into (a) if you allow calculators on the exam. Option (d) is another favorite. When you try to civilize the barbarians, they can still be pretty barbaric. Also, when you live by the mechanical manipulation sword, you die by it (and the death can be gory). Option (e) is an answer Euler might like. Some mathematicians might like it the best. I like it too. But numerically, it's useless. Why, it even needs 4 terms to do better than option (d)! Option (f) is an error that both I and Paul Zorn made in rough drafts of [2] and [4]. I suspect many mathematicians would prefer this answer to (a).

Finally, we come to options (g) and (h). Option (g) is, I presume, the one most mathematicians prefer, I might even say revere. There is mystery and beauty here. There is a long story, a thread leading from circles and geometry to trigonometric identities to the fundamental theorem of calculus. That the area under the curve $y = 1/(1+x^2)$ has something to do with π should provoke awe. The student who answers (g) will have experienced that wonder. Maybe. Probably not. It's just another mechanical manipulation. The answers always come out nice so where is the surprise?

There is still option (h). It is the answer my *HP28S* gives when I apply the PI? button on my USER menu to the number given in option (b). I suspect that option (h) leads to more curiosity, more questions, more respect for mathematics than option (g). Is the answer really $\pi/4$? Let's try more accuracy. Why is it $\pi/4$? How does that PI? button work anyway? By the way, to answer the last question, the PI? button divides the tested number by π and then uses the Euclidean algorithm to estimate the greatest common measure between the given number and the integer 1. More history, more questions! One could also use continued fractions to uncover the fraction behind the 12 digit decimal.

It has been said that exact answers are so precious because they are so infrequent. Yet they are the only answers our students see. Peter Lax complains [1] that calculus classes spend too much time "pulling exact integrals like rabbits out of a hat, and, what is worse, in drilling students how to perform this parlor trick." Another $\pi/4$ pops out of another integral and the audience yawns. I'd be willing to bet, however, that at least one student in a *HP28S* calculus class first learning about Riemann sums will gasp when the PI? button produces option (h).

I would argue that none of these excuses are valid, even the deeper fear of cultural loss. I'm afraid, however, that many will still see technology as just too much trouble. To show you that technology can be just as easy, just as flexible, just as portable, and just as instructive as pencil and paper, I'd like to give a brief demonstration with an *HP28S* calculator. It is a lesson about numerical integration, and it is live–no programs written beforehand, just the bare calculator and me.

First, a subdirectory for numerical integration is created on the USER menu:

'NM.INT' CRDIR

Then we write a program to store an integrand F and limits of integration A and B:

<<'B' STO 'A' STO 'F' STO>> 'FABSTO' STO

We're going to study how well $\int_1^2 (1/x)dx$ is approximated by various rules: Left rectangle, trapezoid, midpoint, Simpson. We enter ' $1/x$ ', 1,2 and press FABSTO. The NM.INT menu now includes the names F, A, B as well as FABSTO. We are going to try different values for N, the number of subdivisions. Here is program that stores a value of N and also stores the subdivision size $(B-A)/N$ in the variable H:

```
<<'N' STO B A - N / 'H' STO >> 'NSTO'
STO
```

Finally, we have the following "workhorse" program, which sums up N values of F at step size H beginning at a given input value, and multiplies the sum by H:

```
<< 'X' STO 0 1 N START F EVAL +
H 'X' STO+ NEXT H * >> 'SUM'STO
```

I won't try to explain this simple program in detail, except to say the the calculator works in reverse Polish taking arguments off the stack. Thus 1 and N before START are the index bounds for the START-NEXT loop. To figure out how the program works, envision the stack before and after each command.

The basic rectangle rules - left, right, and midpoint - are now given by these programs:

```
<< A SUM >> 'LRECT' STO
<< A H + SUM >> 'RRECT' STO
<< A H 2 / + SUM >> 'MID' STO
```

Let's compare LRECT and RRECT for our given integral. First write a program to compute the difference between a computed value and the correct answer, ln 2:

```
<< 2 LN - >> 'ERR' STO
```

Trying $N = 2, 10, 50$ we get the following errors:

	$N=2$	$N=10$	$N=50$
LRECT	.1401...	.0256...	.0050...
RRECT	−.1098...	−.0243...	−.0049...

Clearly, the two rules have errors of approximately equal magnitude but opposite sign. Not quite so obviously, the errors go down by a factor of 5 as N goes up by a factor of 5. Both of these properties of the errors are easily explained with a picture. Note that the linear dependence on H means that each extra decimal digit of accuracy requires an increase of N by a factor of 10. Since $N = 50$ gets two digits of accuracy in 5 seconds, 12 digits of accuracy will take 5×10^{10} seconds or about 1500 years (that's maybe 100,000 sets of batteries).

An intelligent thing to do at this point is to average LRECT and RRECT since their errors apparently cancel. The result is the trapezoid rule:

```
<< LRECT RRECT + 2 / >> 'TRAP' STO
```

Let's compare TRAP and MID errors:

	$N=2$	$N=10$	$N=50$
MID	−.007438...	−.003118...	−.0000124...
TRAP	.015861...	.006242...	.0000249...

The error pattern is clear again. The error of MID is half that of TRAP and has the opposite sign. Also, the errors go down by a factor of around 25 as N increases by a factor of 5.

The next improvement should be obvious: a weighted average of MID and TRAP which cancels errors. The result is Simpson's rule:

```
<< MID 2 * TRAP + 3 / >> 'SIMP' STO
```

The errors for SIMP now go down by a factor of around 625 as N increases by a factor of 5:

	$N=2$	$N=10$	$N=50$
SIMP	.000106...	.000000194...	.000000000307

Since $N = 50$ gets better than 9 digits correct in about 10 seconds and increasing N by a factor of 10 decreases the error by 10^4, we can now get 12 digits of accuracy in a minute with SIMP, a noticeable improvement over 1500 years.

I haven't tried to explain why the midpoint rectangles do so well or how the error depends on the derivatives of the integrand. There are limits to what one can do in 20 minutes, but we have written the programs, tried an example for different rules and different numbers of subdivisions, and uncovered the basic behavior of the errors. This demonstration explains the relationships between the rules and their errors better, I dare say, than all the formulas given in most calculus books. In fact, I suspect few mathematicians would guess that even a CRAY would take an hour or more to compute 12 digits of $\ln 2$ using the left rectangle rule but only a few milliseconds using the midpoint rectangle rule.

It is not by accident that this demonstration did not use the symbolic manipulative powers of this calculator. The truth of the matter is that the HP28S is not a great symbolic algebra system. There will be far more impressive demonstrations of symbolic manipulation later at this conference. *Mathematica*™ could give Euler or Ramanujan a run for the money. But even this little *HP28S* would impress Gauss for its numerical computation, and that is the point. Human beings were not built to do repetitive numerical calculations. That's one reason why we resort to algebra. That's what the Fundamental Theorem of Calculus is for. That's also why we teach so much algebraic manipulation in our calculus classes and so few numerical methods, even though the latter apply effectively and universally and the former do not.

I hope the twilight of paper and pencil means the end of mechanical algebraic manipulation in undergraduate mathematics. Ed Dubinsky at a recent conference on calculus asked some representatives of client disciplines how they would feel if students coming out of mathematics courses could do the simple algebraic manipulation reliably, but ran to a computer whenever the algebra got messy. I claim we should never have let the algebra get that messy in the first place. The only reason we asked students to differentiate $(2+\cos x)^x$ was because it could be done by hand and we could teach students to do it. All the while, we did not teach more important numerical and graphical methods because they could not be done by hand. If our future students are running to the computer with messy algebra, it means we are teaching the wrong things. It means we are asking computers to do things human beings used to be able to do. We should be asking computers to do the things human beings never were able to do.

References

[1] Lax, P. (1986). On the teaching of calculus. In R. Douglas (Ed.), *Toward a Lean and Lively Calculus.* (MAA Notes Number 6). Washington, DC: the Mathematical Association of America.

[2] Tucker, T. (1987, January). Calculators with a college education?, *Focus, 7* No.1 (Jan. 1987).

[3] H.S. Wilf, The disk with the college education, *American Mathematical Monthly, 89*(Jan. 1982), 4-8.

[4] Zorn, P. (1987). Computing in undergraduate mathematics. In L. A. Steen (Ed.), *Calculus for a New Century: A Pump, not a Filter.* (MAA Notes Number 8). Washington, DC: The Mathematical Association of America.