Calculus&*Mathematica*™: Background and future

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The Nationwide Crisis in Calculus Teaching

That calculus teaching is in a crisis is underscored by two national conferences on calculus teaching within the last three years. The first was a large workshop at Tulane University in 1986. This conference stimulated a second conference sponsored by the National Academy of Engineering, the National Academy of Sciences and the National Research Council in late 1987. Robert M. White, President of the National Academy of Engineering said, "Calculus is a critical waystation for the technical manpower this country needs. It must become a pump instead of a filter in the pipeline... We need to teach the calculus in a way that facilitates complex and sophisticated numerical computation in an age of computers. Somehow or other you have to make calculus exciting to the students."

Lynn Arthur Steen, then president of the Mathematical Association of America, bemoaned the fact that at least 80% of the questions asked in typical American calculus exams can be answered by a command key on a (moderately priced) *HP 28S* and went on to ask whether it makes sense to devote most of our calculus to forcing our students to do by hand rote computations that can be better and faster done by computers. Steen predicted, "Mathematics-speaking machines are about to to sweep the campuses... Template exercises and mimicry mathematics, staples of today's texts, will vanish under the assault of computers. By using machines to expedite calculations, students can experience mathematics as it really is – a tentative exploratory discipline in which failures yield clues to success... Weakness (in high school background) will no longer prevent students from pursuing studies that require college math." Perhaps Steen was thinking of the British philosopher and mathematician Alfred North Whitehead who said, "Civilization advances by extending the number of important operations we can peform without thinking about them."

The problem is acute at Illinois. Of the 900,000 American students currently enrolled in calculus, 3500 of them are taking calculus at Illinois. Yet calculus, the fundamental course that teaches scientific and engineering calculations and computations, is unsupported by computers at Illinois. With this evidence, we can take it as a fact that calculus at Illinois and elsewhere needs a new direction and that direction points squarely at integrating the computer into mainline calculus.

But integrating the computer into today's standard calculus course cannot alone solve the basic problems of calculus teaching. The whole course, from foundations to exercises, must be overhauled. The best way to begin the job of writing the calculus course for a new century is to reflect on what calculus is.

The Evolution of American Calculus Courses

Calculus coalesced under the influence of Newton and Leibniz who transformed formerly disjoint procedures into a powerful instrument for systematic calculation. Their revolutionary contribution shook the scientific community of the late 17th century with its almost magical ability to replace many separate ad hoc procedures by a unified procedure. For the next three hundred years, this magnificent intellectual tool solved concrete scientific and technical problems and furnished the framework for the understanding and development of various branches of science.

Until the 1960's, calculus instruction emphasized problem solving. In America, the pre-1960 era saw the heyday of texts such as Granville, Sherwood and Taylor and a few others. During the 1960's, calculus courses gave way to the "new mathematics" of the sixties which influenced mathematics instruction from elementary school through college. It became the fashion to believe that calculus should be offered as a branch of real analysis and the abstract foundations of real analysis began to work their way into calculus books as foundations of calculus. This prompted a shift to what was believed to be "rigor" which pleased most members of the mathematical community until it was realized that the boom in mathematical rigor was a bust for calculus. We are still on the rebound from this era.

The more recent development of calculus has been influenced greatly by the steady expansion of the list of college majors for whom calculus is required. This has spawned an "applications" boom among calculus book publishers and authors rushed to produce texts replete with problems and examples to illustrate how calculus is used in other fields. These texts do not respect the mathematical integrity of calculus but try to justify calculus in terms of other fields of study. In addition the constraints of available textbook space and the very limited student knowledge of these fields of application often make the applications meaningless and pay only lip service to the underlying mathematics in a frantic dash toward application templates and rote procedures.

The reaction we are experiencing today should have been predictable. The essential content of calculus is lost. Expectations concerning mastery have steadily declined. Rote procedures have taken over. Authors protect their image of erudition by writing defensively and by pointing out limits of the theory instead of emphasizing its positive aspects. Many of the calculations discussed in today's texts appear only in calculus texts and have dubious applicability. The students end up not knowing which is more important: underlying principles, unrealistic problems or familiarity with rote procedures for hand computation.

Henri Lebesgue described it best in an essay about French entry level mathematics of the 1930's:

> "The teachers must train their students to answer little fragmentary questions well and they give them model answers that are often veritable masterpieces and that leave no room for criticism. To achieve this, the teachers isolate each question from the whole of mathematics and create for this question alone a perfect language without bothering with its relationship to other questions. Mathematics is no longer a monument but a heap."

Throughout America authors and professors persist in the belief that the foundations of advanced real analysis are the right foundations for calculus. For example, early in the typical calculus text the theorem that guarantees that a continuous function has a maximum and minimum value on each compact interval is stated. The student is told that understanding why this theorem is true is beyond the ability of the student. This message to the student is equivalent to a quick mathematical lobotomy. The student is told that his intuition cannot be trusted. No wonder the student wants to be treated as a machine; we've told the student not to think! Feeling for the positive aspects of calculus is lost and students willingly offer themselves up as robots waiting to be programmed to produce nice solutions to stock problems.

One trouble professors have with this theorem on continuous functions is that this theorem applies in complete generality even to functions that are everywhere continuous but nowhere monotonic. No one ever tells the student about such functions but these functions lurk in the brains of most professors. Lebesgue also addressed this issue:

> "(There) is a real hypocrisy, quite frequent in the teaching of mathematics. The teacher takes verbal precautions which are valued in the sense he gives them but that the sutdents most assuredly will not understand the same way...(many teachers) say with irony 'Fashion dictates precision at one point in the course and all sorts of liberties at another.' The good students have seen enough to be skeptical, too, intead of enthusiastic.... We must attempt an overhaul of the whole structure."

This is the natural result of using foundations that are inappropriately general and abstract. We propose an overhaul of the whole structure from foundations up.

The Right Foundations of Calculus

Here are some key aspects of what we believe are the right foundations and perspectives for our calculus course:

1. The right functions to study in calculus are the piecewise monotone functions. Continuous nowhere monotonic functions are exciting in advanced real analysis. They are useless in calculus. It is clear that any piecewise monotonic function achieves a maximum and minimum value on each compact interval. It is equally clear that a piecewise monotone continuous function has the intermediate value property. Chalk up two victories for the students' intuition.
2. The derivative and integral are used to measure slopes and areas rather than to define slopes and areas. A farmer's field has area even if one border of the field is along a winding river; most farmers can measure this area without the benefit of the abstract definition of the Riemann integral. This view is based on the MAA publication of the notes of Emil Artin's Princeton calculus course of the fifties.

This viewpoint builds confidence and intuition in the mind of the average student. We add that it did not seem to interfere with Hyman Bass's development as a mathematician.

3. Within the context of (1) and (2) above, we proceed with rigor. Within this context the fundamental theorem is easy to prove and understand. Furthermore, formulas for arc length, volumes and the like can be derived in context; within this context, we can expect the students to be responsible for some derivations.
4. Pointwise continuity and pointwise convergence are scrapped in favor of uniform continuity and uniform convergence on compact intervals.

This is also the position taken by Peter Lax and is quite nicely carried off in his Springer UTM calculus course. The reason that we take this view is that uniform convergence and uniform continuity are what the student sees through *Mathematica* plotting. Uniform continuity says simply that for each pixel size, if h is small enough, then the graphs of $f(x+h)$ and $f(x)$ as functions of x are indistinguishable on the screen. Uniform convergence of a sequence to a limits says that for each pixel size the graphs of the late members of the sequence are indistinguishable from the graph of the limit on the screen. What a pleasure it is to see the reaction of students when they watch the difference quotients for $\sin(x)$ converging uniformly to a function whose graph is evidently the graph of $\cos(x)$. Mathematics itself recognizes that uniform convergence is a more elementary idea than pointwise convergence. After all, the uniform limit of a continuous function is continuous, but understanding the possibly pathological behaviour of pointwise limits of continuous functions took the mathematical power of Baire to uncover.

5. The role of approximations is stressed. Knowing that a series converges but not being able to estimate its sum is of dubious value.

Showing that a Taylor series of a function converges at the endpoints without indicating why it converges to the function is a waste of time. In fact the study of series of numbers is not so important as the study of series of functions with estimates on the quality of convergence.

6. Most calculus courses base their study of series of numbers on the axiom that a bounded monotone sequence converges. Operating with this axiom requires a moderate amount of real-variable skill. We replace this axiom with an equivalent axiom:

Given a series, then exactly one of the following two phenomena happens:

(a) The series is convergent.

(b) The partial sums are unbounded or there are numbers a and b with $a < b$ such that infinitely many partial sums are below a and infinitely many partial sums are above b.

This axiom is, of course, equivalent to the axiom that bounded monotone sequences converge. The advantage is that students can understand it and they can work with it. Students at Illinois taught this way in spring 1988 broke the curve on a uniform final in Calculus II.

There is one more point. Students arriving in calculus usually have formed, sometimes in a rather fragile way, a significant foundation of mathematical intuition. Perhaps the most serious mistake made in traditional calculus courses is the destruction of this intuition. The message that students need the intermediate value theorem in order to understand why a cubic polynomial has at least one real root is comforting to the professor and terribly destructive to the students' sets of intuitions. Students who have been working with logarithms for two years and arrive in calculus only to be told that they cannot understand logarithms and exponentials until they see integral calculus emerge confused because their underlying foundation has been damaged. Confidence is lost. Calculus courses and instructors must relinquish the role of acting as curators of dogma and assume the role of builders of mathematical feeling.

Next we give some background on how we plan to integrate *Mathematica* into mainline calculus.

The Appearance of *Mathematica*

Both of us had spent considerable time in the department's curriculum committee on the problems of calculus at the Urbana campus. We both agreed with the nationwide feeling that standard calculus is not successful. On the other hand, we could find no suitable text that did what we and many others thought should be done. As for computers, the only available software was either too primitive or needed exotic expensive computers. Then one day in the winter of 1987-88, we learned of the new *Mathematica* software being developed by Professors Stephen Wolfram, Dan Grayson and Roman Maeder of the UIUC faculty.

Mathematica will do arbitrary precision arithmetic, all manner of symbolic calculations and two and three dimensional plots; it will do all the hand calculations normally associated with calculus (including derivatives, integrals, Newton's method and power series), linear algebra and statistics. It can be used as a calculator or as a programming language and using it is so painless that it blurs the distinction between these two functions. But its real ace in the hole is that it is also a word processor. The word processing capabilities on the Macintosh allow one to create something called "*Mathematica* notebooks." They are live electronic documents which are mixtures of static text and active programs that set up an entirely new vehicle for the teaching of mathematics.

Imagine a calculus book in which every example can be modified on the spot to become infinitely many examples, a calculus book that can plot any calculus function, a calculus book that can differentiate, integrate, find roots, expand in power series – in short a calculus book that can teach the students and act as a slave for them.

This is what we realized one day in the winter of 1987-88. *Mathematica* can be made into an outstanding software package for the teaching of calculus.

One informal session grew into another and some encouraging talks with Stephen Wolfram and Dan Grayson took place. It was not long until we agreed that we would write a calculus course in the form of a *Mathematica* noteboook. We brought in a top notch engineering physics undergraduate student, Don Brown, to help and the three of us have spent the summer and much of this semester on our new course, Calculus&*Mathematica*.

The Electronic Calculus Course, Calculus &*Mathematica*.

The course is a problem course; the text is held to a bare minimum. We believe that the best way for students to involve themselves in calculus is to immerse themselves in a collection of well chosen problems. For this reason, the course proceeds as follows: A snippet of text is followed by a section called Practica of four or five problems. Some of these problems are extensions of the text; others illustrate the utility of the text just studied. This is followed by a collection of problems called Anchors. These problems are chosen to anchor the concepts just learned into the minds of the students.

The whole course takes place on the screen; there is no loss of train of thought in oscillating between the printed page and the computer. The course directs the students to work out concrete exercises and then to embark in their own computational and graphic explorations. This dynamic integration of text and laboratory components of the course is the key feature and it is possible only with *Mathematica*. The computer, the pencil and the text merge into one. Problems are not solved elsewhere but right in their own context. With *Mathematica* handling the rote computations, even ordinary students are able to solve problems far beyond the reach of today's best students. Furthermore, with the computational power of *Mathematica* available at all times, the course can be steered in the direction of underlying concepts and more sophisticated computations than possible in today's courses.

In the course, *Mathematica* is made into a splendid instrument for exposition. For example, students when directed to use *Mathematica* to plot the integrand and the difference quotients for the indefinite integral of the integrand can have little doubt about the meaning of the Fundamental Theorem of Calculus. Students who write a financial planning package should have little doubt about the role that the number e plays in compound interest. Students who have seen the lower Riemann sums fill up the area under the curve of their own choice on a *Mathematica* animation should have little hesitation with the idea of the definite integral as a limit of Riemann sums. Students who have seen the difference quotient converge uniformly to the derivative for the (well-behaved) function of their choice on a *Mathematica* animation should have little doubt about the definition of the derivative. Students who program *Mathematica* animations for the convergence of Taylor and Fourier series should no longer fear these topics the way students of today fear them. Students who can quickly determine that the maximum and minimum values of

$$y = (1-2x)^2(x-x^2)(1-8x+8x^2)^2$$

on $[0,1]$ are 1/64 and 0 respectively should have few problems with generic max–min problems. (This exercise appeared recently in the College Journal of Mathematics as a challenge for professors; its solution using *Mathematica* takes 3 lines and a few seconds.)

In actual problem solutions, C&*M* routes complex calculations through the computer, thus allowing fundamental principles to appear clearly and unhindered by tedious hand calculation or tedious programming. The fundamental principles are given great emphasis simply because a student cannot use *Mathematica* effectively to do calculus unless the fundamental principles are fixed in his or her mind. And the symbolic power of *Mathematica* allows the C&*M* student to handle a more realistic, sophisticated and complex collection of problems than possible by the rote hand methods that plague today's standard calculus course.

The Calculus&*Mathematica* Scientific Advisory Board

Early on it became clear to us that students of C&*M* will be able to solve much more realistic problems that students in the traditional calculus course only can dream of solving. In an attempt to involve the client disciplines in selecting the right problems for C&*M*, we have formed an advisory board of senior scientists on the Illinois faculty, each of whom has an interest in calculus. At this writing, the following colleagues from the University of Illinois at Urbana-Champaign faculty have agreed to share their expertise with us:

- Roy Asford, Professor of Nuclear Engineering
- Donald Carlson, Professor of Theoretical and Applied Mechanics.
- Judith Liebman, Professor of Operations Research, Chancellor for Research and Dean of the Graduate College
- Arthur Robinson, Professor of Civil Engineering
- Nelson Wax, Professor of Electrical Engineering, emeritus
- Stephen Wolfram, Professor of Computer Science, Mathematics and Physics

The Flexibility of Calculus & *Mathematica*

The electronic course C&*M* is completely flexible on two counts. From the student's point of view, the course is flexible because the student can invade the electronic course at almost any point to work problems, write his or her own notes and then save them for later reference or computation. Imagine a printed text in which at the flick of a switch seven blank pages can be made to appear in the middle of the text for student notes, comments or tips. Imagine a printed text that provides right after each problem statement exactly the right amount of space each student needs for the solution. Imagine a calculus course in which most homework assignments are turned in neatly typed and maybe even well written. All of this is routine in C&*M*. Students will appreciate this flexibility, but from the professor's point of view even more flexibility is possible.

All professors have had the experience of looking at a textbook and deciding that the textbook is very good except that the treatment of topic X in Chapter 4 is totally unacceptable. If the textbook is a traditional printed text, the professor goes on to the next book hoping to find something more acceptable. After all, the professor cannot order a book, physically rip out Chapter 4 and replace it with something to the professor's liking. With the electronic course, C&*M*, changing or replacing any chapter, paragraph or exercise is a routine matter. In the classroom, cuts from C&*M* can be made easily for projection on a screen.

Calculus&*Mathematica* and Minority and Physically Handicapped Students

As we mentioned earlier, Lynn Arthur Steen, past president of the Mathematical Association of America said, "Weakness (in high school background) will no longer prevent students from pursuing studies that require college math" because systems like *Mathematica* can shore up students whose background is not ideal. We are going to put this statement to test by committing about forty percent of the student involvement in the initial evaluation of C&*M* to minority students. We have established contact with Uri Treisman of Berkeley who is nationally renowned for his work with minority students at the Berkeley campus. We also have enlisted the help of Paul McCreary of the University of Illinois who is in the process of setting up a Berkeley–like program at Urbana. We are very happy with the enthusiasm with which Treisman and McCreary have greeted our proposal to involve minority students from the start of the The C&*M* project.

Another possibility which will be developed in the future involves physically handicapped students who have lost use of their hands. It was suggested to us that students who cannot write can have satisfactory interface with a minicomputer like a Macintosh. After all, the British physicist and mathematician Stephen Hawking does all his communication via computer. Since the University of Illinois at Urbana-Champaign is renowned for its programs for physically handicapped students, we were intrigued by this suggestion for C&*M*. As a result we made contact with Professor Janet Floyd of the university's Rehabilitation Center. Floyd has expressed a strong interest in pursuing this possibility. Her view is that many more handicapped students would attempt the C&*M* course than can now attempt the traditional calculus course.

The Calculus&*Mathematica* Educational Advisory Board

The C&*M* Educational Advisory board exists to serve several different purposes. It consists of the following members of the Illinois faculty and staff:

- Peter Braunfeld, Professor of Mathematics and Director of Mathematics Education, Department of Mathematics;
- Sandra K. Dawson, Chair, Department of Mathematics, University High School;
- Janet Floyd, Professor of Rehabilitation;
- George K. Francis, Professor of Mathematics;
- Michael Jeffries, Dean and Director of the Office of Minority Students Affairs;
- Anthony L. Peressini, Professor of Mathematics;
- Kenneth Travers, Professor of Secondary Education.

Professor Floyd deals with the physically handicapped students and Mr. Jeffries and his staff deal with the minority student programs. Professor Persessini is a member of the University of Chicago Secondary Mathematics Writing Project and has a keen interest in precalculus mathematics, high school articulation and testing. Professor Francis is known for his work on computer graphics and has operated the highly successful Apple Laboratory at Illinois for eight years. Professor Travers is well known for his leadership role in the Second International Mathematics Study, and will help in the evaluation of C&*M* for high school students. Ms. Dawson will help in evaluating C&*M* for gifted students. Professor Braunfeld, who is the Director of Mathematics Education in the Department of Mathematics, is our main contact for summer teachers' institutes and other educational programs of the Department.

The National Impact of the Course Calculus&*Mathematica*

Even though most of the course exists currently in a preliminary form and has been under development for only nine months, the course C& M has already attracted a lot of nationwide attention. Some examples are:

1. Only nine universities (Yale, Brown, Dartmouth, Boston, Harvard, Cornell, Drexel, NYU, and Illinois) were invited by Apple Computer to the 1988 MacWorld Exposition to demonstrate what they are doing with Macintoshes at their universities. We were the Illinois representatives. We were the only university demonstration that did not feature hypercard.
2. In August, we were invited to demonstrate our course at the National Bureau of Standards in Washington. Over fifty scientists attended our presentation.
3. The University of California, Berkeley invited us to give a demonstration of our course to the assembled faculties of the mathematics and physics departments in November. More than 250 persons attended the presentation.
4. We were invited to demonstrate our course in a one hour presentation at the Conference on Technology in the Teaching of College Mathematics at Ohio State University in October. More than 350 mathematics professors attended our presentation.
5. We were invited by the computer editor for the publication Notices of the American Mathematical Society to publish an article on our course. The article appeared in the November, 1988 issue.
6. The University of Delaware has invited us to demonstrate our course later this winter.
7. Several universities have asked to be test sites for our course. San Francisco, Oklahoma State University and Knox College are going to try to use some of our preliminary material in some sections of their regular calculus course next fall. Missouri is going to test our course in a special laboratory. Northeastern University and Rochester Institute of Technology have also made contact regarding courseware evaluation next fall.

Evaluation of Calculus&*Mathematica*

We will beginevaluation of the C&*M* software in pilot testing at Illinois during the spring 1989 semester. Revisions will be made over the summer and larger scale evaluation and testing of the course will begin at Illinois in the fall 1989 semester. The scope of this testing and evaluation will depend on the number of computers we can muster. Professor Anthony L. Peressini has agreed to lend his considerable skills at testing to lead in evaluating the software. In addition we will make use of the university's Center for Instructional Research and Curriculum Evaluation.

We regard performance in differential equations classes as the ultimate test of a calculus course. The performance in differential equations of all students who take C&*M* at Illinois will be monitored with an eye toward adjustment of the calculus course.

12. Publication of Calculus&*Mathematica*

The published version of C&*M* will involve three separate related products. First is the software which is the focal point of the course. Supporting the software will be two printed books. The first is a book primarily devoted to theory. It will be written in the style of Artin's Princeton course of the fifties. It will present the theory; it will point out clearly what the important parts of the course are and put the secondary parts in perspective. Finally it will be a guide to the software. The second book will be a problem book for students to work on at the computer or away from the computer.

The publisher will be Addison-Wesley's Advanced Books Program. We chose this publisher because Addison-Wesley's Advanced Books Program does not operate through the usual set of calculus-marketing editors. We will be free of the usual marketing considerations that are in large part responsible for the dreary calculus texts which populate today's calculus book market. Plans call for a preliminary edition to be issued in two years followed by a permanent edition later.